

# ABSTRACT COMMENSURATORS OF LATTICES IN LIE GROUPS

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**ABSTRACT.** Let  $\Gamma$  be a lattice in a simply-connected solvable Lie group. We construct a  $\mathbb{Q}$ -defined algebraic group  $\mathcal{A}$  such that the abstract commensurator of  $\Gamma$  is isomorphic to  $\mathcal{A}(\mathbb{Q})$  and  $\text{Aut}(\Gamma)$  is commensurable with  $\mathcal{A}(\mathbb{Z})$ . Our proof uses the algebraic hull construction, due to Mostow, to define an algebraic group  $\mathbf{H}$  so that commensurations of  $\Gamma$  extend to  $\mathbb{Q}$ -defined automorphisms of  $\mathbf{H}$ . We prove an analogous result for lattices in connected linear Lie groups whose semisimple quotient satisfies superrigidity.

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## 1. INTRODUCTION

Given a group  $\Gamma$ , its *abstract commensurator*  $\text{Comm}(\Gamma)$  is the set of equivalence classes of isomorphisms between finite index subgroups of  $\Gamma$ , where two isomorphisms are equivalent if they agree on a finite index subgroup of  $\Gamma$ . Elements of  $\text{Comm}(\Gamma)$  are called *commensurations* of  $\Gamma$ . The abstract commensurator forms a group under composition.

The computation of  $\text{Comm}(\Gamma)$  is a fundamental problem. Commensurations play an important role in the study of rigidity, see e.g. [Mar91], [Zim84]. Commensurations also arise in classification problems in geometry and topology, e.g. [FW05], [FW08], [Avr11], [NR92], [LLR11].

The structure of  $\text{Comm}(\Gamma)$  is often much richer than that of  $\text{Aut}(\Gamma)$ . For example,  $\text{Aut}(\mathbb{Z}^n) \approx \text{GL}_n(\mathbb{Z})$  while  $\text{Comm}(\mathbb{Z}^n) \approx \text{GL}_n(\mathbb{Q})$ . There are a few notable exceptions, which include the cases that  $\Gamma$  is a higher genus mapping class group, that  $\Gamma = \text{Out}(F_n)$  for  $n \geq 4$ , or that  $\Gamma$  is a nonarithmetic lattice in a semisimple Lie group not locally isomorphic to  $\text{PSL}_2(\mathbb{R})$ . In these cases,  $\text{Comm}(\Gamma)$  is virtually isomorphic to  $\Gamma$ ; see [Iva97], [FH07], [Mar91].

This paper is motivated by the following problem.

**Problem.** Let  $G$  be a (connected, linear, real) Lie group and let  $\Gamma \leq G$  be a lattice. Compute  $\text{Comm}(\Gamma)$ .

*Standing Assumption.* Unless otherwise noted, in this paper every Lie group is assumed to be real and connected, and to admit a faithful continuous linear representation. In particular, semisimple Lie groups have finite center.

Every Lie group  $G$  satisfies a short exact sequence

$$1 \rightarrow \text{Rad}(G) \rightarrow G \rightarrow G^{ss} \rightarrow 1,$$

where  $\text{Rad}(G)$  is the maximal connected solvable normal Lie subgroup of  $G$ , and  $G^{ss}$  is semisimple. The study of Lie groups therefore roughly splits into three pieces: one for solvable groups, one for semisimple groups, and a final piece to combine the previous two. Our computation of  $\text{Comm}(\Gamma)$  follows this outline.

**Semisimple  $G$ :** Suppose  $G$  is a connected semisimple Lie group, not locally isomorphic to  $\text{SL}_2(\mathbb{R})$ , and  $\Gamma \leq G$  is an irreducible lattice. Then the computation of  $\text{Comm}(\Gamma)$  is a result of work by Borel, Mostow–Prasad, and Margulis. Recall that the *relative commensurator* of  $\Gamma$  in  $G$  is defined as

$$\text{Comm}_G(\Gamma) = \{g \in G \mid \Gamma \cap g\Gamma g^{-1} \text{ is of finite index in } \Gamma \text{ and } g\Gamma g^{-1}\}.$$

- If  $\Gamma$  is abstractly commensurable to  $\mathbf{G}(\mathbb{Z})$  for some  $\mathbb{Q}$ -defined, adjoint semisimple algebraic group  $\mathbf{G}$  with no  $\mathbb{Q}$ -defined normal subgroup  $\mathbf{N}$  such that  $\mathbf{N}(\mathbb{R})$  is compact, then  $\text{Comm}_{\mathbf{G}}(\Gamma) = \mathbf{G}(\mathbb{Q})$  [Bor66]. (Such a lattice  $\Gamma$  is called *arithmetic*.) For example, if  $\Gamma = \text{PSL}_n(\mathbb{Z})$  for  $n \geq 2$ , then  $\Gamma$  is abstractly commensurable with the group  $\mathbf{G}(\mathbb{Z})$ , where  $\mathbf{G}$  is the  $\mathbb{Q}$ -defined algebraic group  $\mathbf{G} = \text{SL}_n / Z(\text{SL}_n)$ , and so  $\text{Comm}_{\mathbf{G}}(\Gamma) \cong \mathbf{G}(\mathbb{Q})$ . (Note

however that the group  $\mathbf{G}(\mathbb{Q})$  is much larger than  $\mathrm{SL}_n(\mathbb{Q})/Z(\mathrm{SL}_n(\mathbb{Q}))$ , which is sometimes written as  $\mathrm{PSL}_n(\mathbb{Q})$ ; see §7.3 for details.)

- A major theorem of Margulis [Mar91] says that  $\Gamma$  is arithmetic if and only if  $[\mathrm{Comm}_G(\Gamma) : \Gamma] = \infty$ , which occurs if and only if  $\mathrm{Comm}_G(\Gamma)$  is dense in  $G$ .
- If  $G$  has no center and no compact factors, then every commensuration of  $\Gamma$  extends to an automorphism of  $G$  by Mostow–Prasad–Margulis rigidity [Mos73].
- The inner automorphisms of a semisimple real Lie group are finite index in the automorphism group. We therefore have

$$\mathrm{Comm}(\Gamma) \doteq \begin{cases} \mathbf{G}(\mathbb{Q}) & \text{if } \Gamma \text{ is arithmetic} \\ \Gamma & \text{if } \Gamma \text{ is non-arithmetic,} \end{cases}$$

where  $H \doteq K$  if and only if  $H$  and  $K$  are abstractly commensurable, i.e. contain isomorphic finite index subgroups. See Theorem 7.5 for a more precise statement.

*Remark.* In the case  $G = \mathrm{PSL}_2(\mathbb{R})$ , every lattice is either virtually free or virtually the fundamental group of a closed surface group. In either case, the abstract commensurator is not linear; see Proposition 7.6. The abstract commensurator of a surface group has been studied in [Odd05] and [BN00], and may be described as a certain subgroup of the mapping class group of the universal 2-dimensional hyperbolic solenoid.

**Solvable  $G$ :** Suppose  $G$  is a connected, simply-connected solvable real Lie group and  $\Gamma \leq G$  a lattice. In contrast with the semisimple case,  $\mathrm{Aut}(\Gamma)$  is not abstractly commensurable with  $\Gamma$ . On the other hand, the fact that  $\mathrm{Aut}(\Gamma)$  is commensurable with the  $\mathbb{Z}$ -points of a  $\mathbb{Q}$ -defined algebraic group holds for both arithmetic lattices in higher rank semisimple groups and lattices in simply-connected solvable groups. The similarity is reflected in the abstract commensurator.

For example, consider  $G = \mathbb{R}^n$  and  $\Gamma = \mathbb{Z}^n$ . Then  $\mathrm{Aut}(\mathbb{Z}^n) = \mathrm{GL}_n(\mathbb{Z})$  is arithmetic in the  $\mathbb{Q}$ -defined real algebraic group  $\mathrm{Aut}(\mathbb{R}^n) = \mathrm{GL}_n(\mathbb{R})$ , and  $\mathrm{Comm}(\Gamma) = \mathrm{GL}_n(\mathbb{Q})$ . Our first main theorem extends this to lattices in arbitrary simply-connected solvable groups.

**Theorem 1.1.** *Let  $\Gamma$  be a lattice in a connected, simply-connected solvable Lie group  $G$ . Then there is some  $\mathbb{Q}$ -defined algebraic group  $\mathcal{A}_\Gamma$  such that*

$$\mathrm{Comm}(\Gamma) \approx \mathcal{A}_\Gamma(\mathbb{Q})$$

*and the image of  $\mathrm{Aut}(\Gamma)$  in  $\mathcal{A}_\Gamma(\mathbb{Q})$  is commensurable with  $\mathcal{A}_\Gamma(\mathbb{Z})$ .*

*Remark.* If  $G$  is nilpotent then  $\mathcal{A}_\Gamma(\mathbb{R}) = \mathrm{Aut}(G)$ ; see Theorem 4.2.

A fundamental difficulty in dealing with lattices in solvable groups is lack of rigidity; automorphisms of a lattice may not extend to automorphisms of its ambient Lie group, even virtually. There are a number of results

addressing this to some extent, most notably [Wit95]. Instead of applying results providing rigidity in the ambient Lie group, our proof of Theorem 1.1 utilizes the *virtual algebraic hull*, a connected solvable  $\mathbb{Q}$ -defined algebraic group  $\mathbf{H}$  in which  $\Gamma$  virtually embeds as a Zariski-dense subgroup. There is a natural map

$$\xi : \text{Comm}(\Gamma) \rightarrow \text{Aut}(\mathbf{H})$$

such that  $\xi([\phi])$  is  $\mathbb{Q}$ -defined for each  $[\phi] \in \text{Comm}(\Gamma)$ . The automorphism group  $\text{Aut}(\mathbf{H})$  naturally has the structure of a  $\mathbb{Q}$ -defined algebraic group, and we set  $\mathcal{A}$  equal to the Zariski-closure of  $\xi(\text{Comm}(\Gamma))$  in  $\text{Aut}(\mathbf{H})$ .

A finite index subgroup of  $\text{Comm}(\Gamma)$  can be understood fairly concretely.  $\Gamma$  has a unique maximal normal nilpotent subgroup  $\text{Fitt}(\Gamma) \leq \Gamma$ . Let  $\mathbf{F}$  denote the Zariski-closure of  $\text{Fitt}(\Gamma)$  in  $\mathbf{H}$ . Define  $\text{Comm}_{\mathbf{H}|\mathbf{F}}(\Gamma)$  to be the group of commensurations trivial on  $\Gamma/\text{Fitt}(\Gamma)$ . By rigidity of tori,  $\text{Comm}_{\mathbf{H}|\mathbf{F}}$  is of finite index in  $\text{Comm}(\Gamma)$ . The group  $\text{Comm}_{\mathbf{H}|\mathbf{F}}$  decomposes as the product of the group of commensurations arising from conjugation by elements of  $\mathbf{F}(\mathbb{Q})$  and the group of commensurations fixing a maximal  $\mathbb{Q}$ -defined torus  $\mathbf{T} \leq \mathbf{H}$ . See §5.5 and §6 for details.

*Remark.* Mostow’s algebraic hull construction is extended to certain virtually polycyclic groups  $\Gamma$  in [Bau04], and this hull is applied in [BG06] to describe  $\text{Aut}(\Gamma)$  and  $\text{Out}(\Gamma)$ . Our proof of Theorem 1.1 is heavily based on the techniques in [BG06]. However, the abstract commensurator only depends on  $\Gamma$  up to finite index, and therefore requires less attention to delicate structure of  $\Gamma$ .

**General  $G$ :** When  $G$  is not necessarily either semisimple or solvable, we prove:

**Theorem 1.2.** *Suppose  $G$  is a connected, linear Lie group with simply-connected solvable radical. Suppose  $\Gamma \leq G$  is a lattice with the property that there is no surjection  $\phi : G \rightarrow H$  to any group  $H$  locally isomorphic to any  $\text{SO}(1, n)$  or  $\text{SU}(1, n)$  so that  $\phi(\Gamma)$  is a lattice in  $H$ . Then*

- (1)  $\Gamma$  virtually embeds in a  $\mathbb{Q}$ -defined algebraic group  $\mathbf{G}$  with Zariski-dense image so that every commensuration  $[\phi] \in \text{Comm}(\Gamma)$  induces a unique  $\mathbb{Q}$ -defined automorphism of  $\mathbf{G}$  virtually extending  $\phi$ .
- (2) There is a  $\mathbb{Q}$ -defined algebraic group  $\mathcal{B}$  so that

$$\text{Comm}(\Gamma) \approx \mathcal{B}(\mathbb{Q})$$

*and the image of  $\text{Aut}(\Gamma)$  in  $\mathcal{B}$  is commensurable with  $\mathcal{B}(\mathbb{Z})$ .*

The group  $\mathbf{G}$  of Theorem 1.2 is, roughly speaking, constructed as the semidirect product of the virtual algebraic hull  $\mathbf{H}$  of the “solvable part” of  $\Gamma$  and a semisimple group  $\mathbf{S}$  such that  $\mathbf{S}(\mathbb{Z})$  is commensurable with the “semisimple part” of  $\Gamma$ . The technical work comes first in making this precise, and second in constructing an action of  $\mathbf{S}$  on  $\mathbf{H}$  compatible with the group structure of  $\Gamma$ .

In the case that  $\Gamma$  surjects to a non-superrigid lattice, commensurations do not generally extend to automorphisms of  $\mathbf{G}$ . The hypothesis that  $\Gamma$  does not surject to a lattice in either  $\mathrm{SO}(1, n)$  or  $\mathrm{SU}(1, n)$  is used to apply the superrigidity results of Margulis and Corlette, which are used to extend commensurations of  $\Gamma$  to automorphisms of  $\mathbf{G}$ . Additional commensurations arise from the nontriviality of  $H^1(\Gamma, \mathbb{Q})$ ; see §8.

*Remark.* If  $\mathbf{A}$  is a  $\mathbb{Q}$ -defined algebraic group, then there is a natural map  $\Xi : \mathrm{Aut}_{\mathbb{Q}}(\mathbf{A}) \rightarrow \mathrm{Comm}(\mathbf{A}(\mathbb{Z}))$ . If  $\mathbf{A}$  is unipotent, or if  $\mathbf{A}$  is  $\mathbb{Q}$ -simple, semisimple, and such that  $\mathbf{A}(\mathbb{R})$  is not compact and has no factor isogenous to  $\mathrm{PSL}_2(\mathbb{R})$ , then  $\Xi$  is injective because  $\mathbf{A}(\mathbb{Z})$  is Zariski-dense in  $\mathbf{A}$ , and  $\Xi$  is surjective because  $\mathbf{A}(\mathbb{Z})$  is strongly rigid in  $\mathbf{A}$  by results of Malcev and Mostow–Prasad–Margulis (see Theorems 4.2 and 7.3). See [GP99b] for similar results for solvable arithmetic groups.

The difficulty in proving our results comes from the fact that lattices in solvable Lie groups need not be commensurable with the  $\mathbb{Z}$ -points of any algebraic group; see [Seg83] for an example. When  $\Gamma$  is a lattice in a simply-connected solvable group, the algebraic hull construction provides an algebraic group  $\mathbf{H}$  so that  $\Gamma$  virtually embeds in  $\mathbf{H}(\mathbb{Z})$ , but in general  $\Gamma$  is too small to be commensurable with  $\mathbf{H}(\mathbb{Z})$ . Despite this, automorphisms of  $\mathbf{H}$  extending commensurations of  $\Gamma$  may readily be understood in terms of the algebraic structure of  $\mathbf{H}$ .

**Outline:** We review basic results in the theory of linear algebraic groups in §2. We define and review basic properties of the abstract commensurator in §3, including definitions of commensuristic and strongly commensuristic subgroups.

In §4 we prove Theorem 1.1 for nilpotent  $G$  using classical rigidity of nilpotent lattices. In §5, we review the basic theory of polycyclic groups and the definition of the algebraic hull. Our exposition largely follows [BG06]. We define the unipotent shadow, and discuss the algebraic structure of  $\mathrm{Aut}(\mathbf{H})$ . In §6 we prove Theorem 1.1.

In §7 we review results on commensurations of lattices in semisimple Lie groups, which are due primarily to Borel, Margulis, and Mostow–Prasad. In §8 we combine the solvable and semisimple cases to prove Theorem 1.2.

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## 2. NOTATION AND PRELIMINARIES

Let  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}$ , and  $\mathbb{C}$  denote the integer, rational, real, and complex numbers, respectively. Let  $K \subseteq \mathbb{C}$  be a subfield. A group  $\Gamma$  *virtually* has a property  $P$  if there is a finite index subgroup of  $\Gamma$  with  $P$ . In particular, if  $\Gamma \leq G$ , say that a homomorphism  $\phi : \Gamma \rightarrow H$  *virtually* extends to a homomorphism  $\Phi : G \rightarrow H$  if there is a finite index subgroup  $\Gamma_0 \leq \Gamma$  so that  $\phi|_{\Gamma_0} = \Phi|_{\Gamma_0}$ .

**2.1. Algebraic groups.** We use the basic theory of linear algebraic groups. A good general reference is [Bor91]. Our preliminaries overlap with those in [BG06].

An *algebraic group*  $\mathbf{A}$  is a subset of  $\mathrm{GL}_n(\mathbb{C})$  for some natural number  $n$  that is closed in the Zariski topology. An algebraic group  $\mathbf{A}$  is *K-defined* if it is closed in the Zariski topology with closed subsets those defined by polynomials with coefficients in  $K$ . A *K-defined* algebraic group is called a *K-group*. A *K-group* is *K-simple* if it has no connected normal *K-defined* subgroup, and *absolutely simple* if it has no connected normal subgroup defined over  $\mathbb{C}$ . (Such groups are sometimes called “almost *K-simple*” or “absolutely almost simple,” respectively.)

If  $R$  is a subring of  $\mathbb{C}$ , then define  $\mathbf{A}(R) = \mathbf{A} \cap \mathrm{GL}_n(R) \subseteq \mathrm{GL}_n(\mathbb{C})$ . If  $V$  is a complex vector space with a fixed basis, then  $V(R)$  denotes the collection of  $R$ -linear combinations of basis vectors. Every algebraic group has finitely many Zariski-connected components. The connected component of the identity  $\mathbf{A}^0$  is a finite index subgroup of  $\mathbf{A}$ .

**Proposition 2.1** (cf. [Bor91, 1.3]). *If  $\mathbf{A}$  is *K-defined* and  $\Gamma \leq \mathbf{A}(K)$  is a subgroup, then the Zariski-closure of  $\Gamma$  is a *K-defined* subgroup.*

**Proposition 2.2** (cf. [Bor91, 18.3]). *If  $\mathbf{A}$  is a connected *K-defined* algebraic group, then  $\mathbf{A}(K)$  is Zariski-dense in  $\mathbf{A}$ .*

A *homomorphism of algebraic groups* is a group homomorphism that is also a morphism of the underlying affine algebraic variety. If both varieties are *K-defined* and the variety morphism is defined over  $K$ , then we say that the homomorphism of algebraic groups is *K-defined*. A *K-defined isomorphism* is a *K-defined* morphism of algebraic groups with an inverse that is also *K-defined*. Let  $\mathrm{Aut}(\mathbf{A})$  denote the group of automorphisms of  $\mathbf{A}$  as an algebraic group, and  $\mathrm{Aut}_K(\mathbf{A})$  denote the group of *K-defined* automorphisms of  $\mathbf{A}$ .

Quotients and semi-direct products of *K-defined* algebraic groups exist:

**Lemma 2.3** (cf. [Bor91, 6.8]). *Suppose  $G$  is a *K-defined* algebraic group and  $H \leq G$  is a normal, closed, *K-defined* subgroup. Then  $G/H$  is a *K-defined* algebraic group, and the quotient map  $\pi : G \rightarrow G/H$  is *K-defined*.*

**Lemma 2.4** (cf. [Bor91, 1.11]). *Suppose  $G$  and  $H$  are *K-defined* algebraic groups. Suppose  $G$  acts on  $H$ , and the action map  $\alpha : G \times H \rightarrow H$  is*

$K$ -defined. Then the semi-direct product  $H \rtimes G$  naturally has the structure of a  $K$ -defined algebraic group.

A torus is an algebraic group isomorphic to  $(\mathbb{C}^*)^n$  for some  $n$ . Because the automorphism group of a torus is discrete, we have:

**Lemma 2.5** (cf. [Bor91, 8.10]). *Let  $\mathbf{T}$  be any torus and  $\mathbf{A}$  any algebraic group acting on  $\mathbf{T}$  by homomorphisms, so that the map  $\mathbf{A} \times \mathbf{T} \rightarrow \mathbf{T}$  is a morphism of varieties. Then  $\mathbf{A}^0$  acts trivially on  $\mathbf{T}$ .*

Let  $\mathbf{A}$  be a  $K$ -defined algebraic group. The *unipotent radical*  $\mathbf{U}_{\mathbf{A}}$  of  $\mathbf{A}$  is the unique maximal closed unipotent normal subgroup of  $\mathbf{A}$ . The *solvable radical*  $\text{Rad}(\mathbf{A})$  of  $\mathbf{A}$  is unique maximal connected closed solvable normal subgroup of  $\mathbf{A}$ . Both  $\mathbf{U}_{\mathbf{A}}$  and  $\text{Rad}(\mathbf{A})$  are  $K$ -defined subgroups of  $\mathbf{A}$ . Say  $\mathbf{A}$  is *reductive* if  $\mathbf{U}_{\mathbf{A}}$  is trivial, and *semisimple* if  $\text{Rad}(\mathbf{A})$  is trivial. A *Levi subgroup* is a connected reductive subgroup  $\mathbf{L} \leq \mathbf{A}$  so that  $\mathbf{A} = \mathbf{U}_{\mathbf{A}} \rtimes \mathbf{L}$ .

**Theorem 2.6** (Mostow, see [PR94, Theorem 2.3]). *For any  $K$ -defined algebraic group  $\mathbf{A}$ , there is a  $K$ -defined Levi subgroup  $\mathbf{L}$ . Moreover, any reductive  $K$ -defined subgroup is conjugate by an element of  $\mathbf{U}_{\mathbf{A}}(K)$  into  $\mathbf{L}$ .*

The following summarizes some standard results concerning solvable algebraic groups.

**Proposition 2.7** (cf. [Bor91, 10.6]). *Let  $\mathbf{H}$  be a  $\mathbb{Q}$ -defined connected solvable algebraic group. Then:*

- (1)  $\mathbf{U}_{\mathbf{H}}$  consists of all unipotent elements of  $\mathbf{H}$ .
- (2)  $[\mathbf{H}, \mathbf{H}] \subseteq \mathbf{U}_{\mathbf{H}}$ .
- (3) There is a  $\mathbb{Q}$ -defined maximal torus  $\mathbf{T} \leq \mathbf{H}$ .
- (4) Any two maximal  $\mathbb{Q}$ -defined tori are conjugate by an element of  $[\mathbf{H}, \mathbf{H}](\mathbb{Q})$ .
- (5) If  $\mathbf{T}$  is a  $\mathbb{Q}$ -defined maximal torus, then  $\mathbf{H}$  is a semidirect product  $\mathbf{H} = \mathbf{U}_{\mathbf{H}} \rtimes \mathbf{T}$ .
- (6) If  $\mathbf{D}$  is the centralizer of a maximal torus and  $\mathbf{F} \leq \mathbf{U}_{\mathbf{H}}$  is any normal subgroup containing  $[\mathbf{H}, \mathbf{H}]$ , then  $\mathbf{H} = \mathbf{F} \cdot \mathbf{D}$ .

**2.2. Semisimple Lie and algebraic groups.** A general reference for the theory of semisimple algebraic groups used here is [Mar91, Chapter 1].

If  $\mathbf{A}$  is an  $\mathbb{R}$ -defined algebraic group, then  $\mathbf{A}(\mathbb{R})$  is a real Lie group with finitely many connected components. We always consider  $\mathbf{A}(\mathbb{R})$  with its topology as a Lie group. In particular,  $\mathbf{A}(\mathbb{R})^0$  denotes the connected component of the identity in the Lie group topology.

**Proposition 2.8** ([Zim84]). *Suppose  $S$  is a connected semisimple Lie group with trivial center. Then there is a  $\mathbb{Q}$ -defined semisimple algebraic group  $\mathbf{S}$  so that  $S = \mathbf{S}(\mathbb{R})^0$ .*

An *isogeny* of algebraic groups is a surjective morphism with finite kernel. An isogeny is *central* if its kernel is central. A connected semisimple algebraic

group  $\mathbf{S}$  is *simply-connected* if every central isogeny  $\Phi : \mathbf{S}' \rightarrow \mathbf{S}$  is an isomorphism. For every connected  $K$ -defined semisimple algebraic group  $\mathbf{S}$ , there is a unique simply-connected  $K$ -defined semisimple algebraic group  $\tilde{\mathbf{S}}$  and central  $K$ -defined isogeny  $p : \tilde{\mathbf{S}} \rightarrow \mathbf{S}$ . Every simply-connected semisimple  $K$ -group decomposes uniquely into a product of  $K$ -simple simply-connected  $K$ -groups.

**Proposition 2.9** (c.f. [Mar91, I.2.6.5]). *Suppose  $\mathbf{A}$  is an  $\mathbb{R}$ -defined algebraic group, and  $\mathbf{S}$  is a simply-connected semisimple  $\mathbb{R}$ -defined algebraic group. Let  $\rho : \mathbf{S}(\mathbb{R})^0 \rightarrow \mathbf{A}(\mathbb{R})$  be a continuous representation. Then  $\rho$  extends to an  $\mathbb{R}$ -defined morphism  $\tilde{\rho} : \mathbf{S} \rightarrow \mathbf{A}$ .*

A  $\mathbb{Q}$ -defined semisimple algebraic group  $\mathbf{S}$  is *without  $\mathbb{Q}$ -compact factors* if there is no nontrivial  $\mathbb{Q}$ -defined connected normal subgroup  $\mathbf{N} \leq \mathbf{S}$  such that  $\mathbf{N}(\mathbb{R})$  is compact. (This terminology is not standard.)

**Theorem 2.10** (Borel Density Theorem [Bor66]). *Suppose  $\mathbf{S}$  is a connected,  $\mathbb{Q}$ -defined semisimple algebraic group without  $\mathbb{Q}$ -compact factors. Then  $\mathbf{S}(\mathbb{Z})$  is Zariski-dense in  $\mathbf{S}$ .*

**Definition 2.11.** Let  $\mathbf{S}$  be an  $\mathbb{R}$ -defined semisimple algebraic group. The *real rank* of  $\mathbf{S}$ , denoted  $\mathbb{R}\text{-rank}(\mathbf{S})$ , is the maximal dimension of an abelian  $\mathbb{R}$ -defined subgroup diagonalizable over  $\mathbb{R}$ . If  $S$  is a connected semisimple Lie group with finite center, define  $\mathbb{R}\text{-rank}(S)$  to be the real rank of the  $\mathbb{Q}$ -group  $\mathbf{S}$  satisfying  $\mathbf{S}(\mathbb{R})^0 = S/Z(S)$ .

Our results use strong rigidity of Mostow, Prasad, and Margulis, and superrigidity results of Margulis, Corlette, and Gromov–Schoen. The following statement is an immediate corollary of [GP99a, 2.6].

**Theorem 2.12** (c.f. [GP99a]). *Suppose  $\mathbf{S}_1$  and  $\mathbf{S}_2$  are connected, simply-connected,  $\mathbb{Q}$ -defined,  $\mathbb{Q}$ -simple semisimple algebraic groups with  $\mathbb{R}\text{-rank}(\mathbf{S}_1) > 0$  and  $\mathbb{R}\text{-rank}(\mathbf{S}_2) > 0$ . Suppose  $\Gamma_1$  and  $\Gamma_2$  are finite index subgroups of  $\mathbf{S}_1(\mathbb{Z})$  and  $\mathbf{S}_2(\mathbb{Z})$ , respectively. Assume that  $\mathbf{S}_1(\mathbb{R})^0$  has no simple factor locally isomorphic to  $\mathrm{SL}_2(\mathbb{R})$  such that the projection of  $\Gamma_1 \cap \mathbf{S}_1(\mathbb{R})^0$  into this factor is discrete. Then every isomorphism  $\Gamma_1 \rightarrow \Gamma_2$  virtually extends to a  $\mathbb{Q}$ -defined isomorphism of algebraic groups  $\mathbf{S}_1 \rightarrow \mathbf{S}_2$ .*

### 3. THE ABSTRACT COMMENSURATOR

Let  $\Gamma$  be an abstract group. In this section we will define the abstract commensurator  $\mathrm{Comm}(\Gamma)$  and review its basic properties.

A *partial automorphism* of  $\Gamma$  is an isomorphism  $\phi : \Gamma_1 \rightarrow \Gamma_2$  where  $\Gamma_1$  and  $\Gamma_2$  are finite index subgroups of  $\Gamma$ . Two partial automorphisms  $\phi$  and  $\phi'$  of  $\Gamma$  are *equivalent* if there is some finite index subgroup  $\Gamma_3 \leq \Gamma$  so that  $\phi$  and  $\phi'$  are both defined on  $\Gamma_3$  and  $\phi|_{\Gamma_3} = \phi'|_{\Gamma_3}$ . If  $\phi : \Gamma_1 \rightarrow \Gamma_2$  is a partial automorphism of  $\Gamma$ , its equivalence class  $[\phi]$  is called a *commensuration* of



$\Gamma$ . There is a natural composition of commensurations. If  $\phi : \Gamma_1 \rightarrow \Gamma_2$  and  $\phi' : \Gamma'_1 \rightarrow \Gamma'_2$  are partial automorphisms of  $\Gamma$ , then we define

$$[\phi'] \circ [\phi] = [\phi' \circ \phi|_{\phi^{-1}(\Gamma_2 \cap \Gamma'_1)}].$$

This definition is independent of choice of representatives of equivalence classes  $[\phi]$  and  $[\phi']$ .

**Definition 3.1.** Given a group  $\Gamma$ , the *abstract commensurator*  $\text{Comm}(\Gamma)$  is the group of commensurations of  $\Gamma$  under composition.

**Example 3.2.**  $\text{Comm}(\mathbb{Z}^n) \approx \text{GL}_n(\mathbb{Q})$

Two subgroups  $\Delta_1, \Delta_2 \leq \Gamma$  are *commensurable* if  $[\Delta_1 : \Delta_1 \cap \Delta_2] < \infty$  and  $[\Delta_2 : \Delta_1 \cap \Delta_2] < \infty$ . Define an equivalence relation on the set of subgroups of  $\Gamma$  by  $\Delta_1 \sim \Delta_2$  if and only if  $\Delta_1$  and  $\Delta_2$  are commensurable. Let  $[\Delta]$  denote the equivalence class of a subgroup  $\Delta \leq \Gamma$  under this relation. The abstract commensurator  $\text{Comm}(\Gamma)$  acts on the set of commensurability classes of subgroups of  $\Gamma$  in an obvious way; given a partial automorphism  $\phi : \Gamma_1 \rightarrow \Gamma_2$  of  $\Gamma$ , define

$$[\phi] \cdot [\Delta] = [\phi(\Delta \cap \Gamma_1)].$$

Clearly this is independent of choice of representatives  $\phi$  and  $\Delta$ .

**Definition 3.3** (Commensuristic subgroup). A subgroup  $\Delta \leq \Gamma$  is *commensuristic* if  $[\phi] \cdot [\Delta] = [\Delta]$  for every  $[\phi] \in \text{Comm}(\Gamma)$ . A subgroup  $\Lambda \leq \Gamma$  is *strongly commensuristic* if, for every partial automorphism  $\phi : \Gamma_1 \rightarrow \Gamma_2$  of  $\Gamma$ ,

$$\phi(\Gamma_1 \cap \Lambda) = \Gamma_2 \cap \Lambda.$$

Every strongly commensuristic subgroup is both characteristic and commensuristic. Neither converse holds.

**Example 3.4.** Consider the group

$$\Gamma = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y \in \mathbb{Z} \text{ and } z \in \frac{1}{2}\mathbb{Z} \right\} \leq \text{GL}_3(\mathbb{Z}).$$

Note that  $\Gamma$  is a lattice in the real Heisenberg group. Denote elements of  $\Gamma$  by triples  $(x, y, z)$  where  $x, y$ , and  $z$  are as above. The center  $Z(\Gamma)$  is infinite cyclic, generated by  $(0, 0, \frac{1}{2})$ , and contains the commutator subgroup  $[\Gamma, \Gamma]$  with index 2. By Proposition 4.1, the center  $Z(\Gamma)$  is strongly commensuristic and the commutator subgroup  $[\Gamma, \Gamma]$  is commensuristic. Further,  $[\Gamma, \Gamma]$  is evidently characteristic.

Now consider the subgroup  $\Gamma_2 \leq \Gamma$  generated by  $(2, 0, 0)$ ,  $(0, 2, 0)$ , and  $(0, 0, 2)$ . Then the map  $\phi : \Gamma \rightarrow \Gamma_2$  defined by  $\phi(x, y, z) = (2x, 2y, 4z)$  is a partial automorphism of  $\Gamma$ . But  $\phi$  takes  $[\Gamma, \Gamma]$  to  $[\Gamma_2, \Gamma_2]$ , which is the infinite cyclic group generated by  $(0, 0, 4)$ . Therefore  $[\Gamma, \Gamma]$  is not strongly commensuristic.  $\square$

*Question.* Let  $\Gamma$  be a finitely generated group. Is every characteristic subgroup of  $\Gamma$  commensuristic? Is every commensuristic subgroup of  $\Gamma$  commensurable with a characteristic subgroup? Is every commensuristic subgroup of  $\Gamma$  commensurable with a strongly commensuristic subgroup?

The notions of ‘commensuristic’ and ‘strongly commensuristic’ are motivated by the following lemma.

**Lemma 3.5.** *If  $\Delta \leq \Gamma$  is commensuristic, then restriction induces a homomorphism*

$$\text{Comm}(\Gamma) \rightarrow \text{Comm}(\Delta).$$

*If  $\Delta$  is normal in  $\Gamma$  and strongly commensuristic, then there is a homomorphism*

$$\text{Comm}(\Gamma) \rightarrow \text{Comm}(\Gamma/\Delta).$$

*Proof.* Suppose  $\Delta \leq \Gamma$  is commensuristic. Let  $\phi : \Gamma_1 \rightarrow \Gamma_2$  be a partial automorphism of  $\Gamma$ . Then  $\phi(\Delta \cap \Gamma_1)$  is commensurable with  $\Delta$ , and so  $\Delta_1 = \phi^{-1}(\Delta \cap \phi(\Delta \cap \Gamma_1))$  is a finite index subgroup of  $\Delta$ . The restriction of  $\phi$  to  $\Delta_1$  defines a partial automorphism of  $\Delta$ . Restriction clearly respects the equivalence relation on partial automorphisms and is compatible with composition, so this determines a well-defined homomorphism  $\text{Comm}(\Gamma) \rightarrow \text{Comm}(\Delta)$ .

Suppose now that  $\Delta \leq \Gamma$  is strongly commensuristic and normal, and let  $\phi : \Gamma_1 \rightarrow \Gamma_2$  be a partial automorphism of  $\Gamma$ . Then  $\phi$  descends a map  $\hat{\phi} : \Gamma_1 \rightarrow \Gamma_2/(\Gamma_2 \cap \Delta)$ . Because  $\Delta$  is strongly commensuristic, the kernel of this map is precisely  $\Gamma_1 \cap \Delta$ . There is then an isomorphism

$$\phi_* : \Gamma_1/(\Gamma_1 \cap \Delta) \rightarrow \Gamma_2/(\Gamma_2 \cap \Delta).$$

The map  $\phi_*$  is a partial automorphism of  $\Gamma/\Delta$ . If  $\phi_1$  and  $\phi_2$  are equivalent partial automorphisms, then  $\hat{\phi}_1$  and  $\hat{\phi}_2$  agree on some finite index subgroup of  $\Gamma_1$ . It follows that  $(\phi_1)_*$  and  $(\phi_2)_*$  are equivalent partial automorphisms of  $\Gamma/\Delta$ . Therefore there is a well-defined map  $\text{Comm}(\Gamma) \rightarrow \text{Comm}(\Gamma/\Delta)$ , which is obviously a homomorphism.  $\square$

*Remark.* Lemma 3.5 is implicitly applied in [LM06], using that the center of the braid group is strongly commensuristic.

We will often use the following corollaries implicitly in this paper.

**Corollary 3.6.** *If  $[\Gamma : \Gamma'] < \infty$  then  $\text{Comm}(\Gamma') \approx \text{Comm}(\Gamma)$ .*

*Proof.* Every finite index subgroup of  $\Gamma$  is commensurable with  $\Gamma$ , so  $\Gamma'$  is commensuristic. The induced restriction map  $r : \text{Comm}(\Gamma) \rightarrow \text{Comm}(\Gamma')$  is clearly injective. On the other hand,  $r$  is surjective because any finite index subgroup of  $\Gamma'$  is a finite index subgroup of  $\Gamma$ .  $\square$

Two groups  $\Gamma$  and  $\Lambda$  are called *abstractly commensurable*, written  $\Gamma \doteq \Lambda$ , if there are finite index subgroups  $\Gamma_1 \leq \Gamma$  and  $\Lambda_1 \leq \Lambda$  such that  $\Gamma_1 \approx \Lambda_1$ .

**Corollary 3.7.** *If  $\Gamma \doteq \Lambda$  then  $\text{Comm}(\Gamma) \approx \text{Comm}(\Lambda)$ .*  $\square$

There is a weaker notion of equivalence similar to that of abstract commensurability. Define a relation on groups by  $\Gamma_1 \sim \Gamma_2$  if there is a homomorphism  $\phi : \Gamma_1 \rightarrow \Gamma_2$  with finite index image and finite kernel. Say that  $\Gamma_1$  and  $\Gamma_2$  are *commensurable up to finite kernels* if they lie in the same equivalence class of the equivalence relation generated by  $\sim$ . In general, groups which are commensurable up to finite kernels need not be abstractly commensurable.

Recall that a group  $\Gamma$  is *residually finite* if the intersection of all finite index subgroups is trivial. It is a theorem of Malcev that finitely generated linear groups are residually finite. The following is an easy exercise; see [dlH00] for proof.

**Proposition 3.8.** *Two residually finite groups are abstractly commensurable if and only if they are commensurable up to finite kernels.*

#### 4. COMMENSURATIONS OF LATTICES IN NILPOTENT GROUPS

**4.1. Example: the Heisenberg group.** Consider the  $(2n+1)$ -dimensional Heisenberg group

$$\mathcal{H}^{2n+1} = \left\{ \begin{pmatrix} 1 & \mathbf{x} & z \\ 0 & I_n & \mathbf{y} \\ 0 & 0 & 1 \end{pmatrix} \mid \mathbf{x}, \mathbf{y}^t \in \mathbb{C}^n \text{ and } z \in \mathbb{C} \right\} \leq \mathrm{GL}_{n+2}(\mathbb{C}).$$

Then  $N = \mathcal{H}^{2n+1}(\mathbb{R})$  is a simply-connected, 2-step nilpotent Lie group in which  $\Gamma = \mathcal{H}^{2n+1}(\mathbb{Z})$  is a lattice. Let  $Z = Z(N)$  denote the center of  $N$ ; note that  $Z \approx \mathbb{R}$  and that  $N/Z \approx \mathbb{R}^{2n}$ . Introduce coordinates on  $N$  by writing a matrix as above as the pair  $(v, t)$ , where  $v = (\mathbf{x}, \mathbf{y}^t) \in \mathbb{R}^{2n}$  and  $t = z \in \mathbb{R}$ . One can check that, in this notation,

$$[(u, s), (v, t)] = (0, \omega(u, v)),$$

where  $\omega$  is the standard symplectic form on  $\mathbb{R}^{2n}$ .

Suppose  $\phi : \Gamma_1 \rightarrow \Gamma_2$  is a partial automorphism of  $\Gamma$ . We will see that  $\phi(\Gamma_1 \cap Z) = \Gamma_2 \cap Z$ , and so  $[\phi]$  induces a commensuration  $[\bar{\phi}]$  of  $\Gamma/Z(\Gamma) \approx \mathbb{Z}^{2n}$ . By the condition

$$[0, \omega(\bar{\phi}(u), \bar{\phi}(v))] = [\phi(u, s), \phi(v, t)] = \phi([(u, s), (v, t)]) = (0, \bar{\phi}(\omega(u, v)))$$

the induced map  $\bar{\phi} \in \mathrm{GL}_{2n}(\mathbb{Q})$  must in fact belong to the general symplectic group  $\mathrm{GSp}_{2n}(\mathbb{Q})$ , which is defined as

$$\mathrm{GSp}_{2n}(\mathbb{Q}) = \{A \in \mathrm{GL}_{2n}(\mathbb{Q}) \mid \omega(Au, Av) = \alpha \omega(u, v) \text{ for some } \alpha \in \mathbb{Q}^*\}.$$

In fact the induced map  $\Theta : \mathrm{Comm}(\Gamma) \rightarrow \mathrm{GSp}_{2n}(\mathbb{Q})$  is surjective. Each partial automorphism  $\phi : \Gamma_1 \rightarrow \Gamma_2$  such that  $[\phi] \in \ker(\Theta)$  is trivial on  $Z$ , hence determined by an element of  $H^1(\pi(\Gamma_1), \mathbb{Z})$ , where  $\pi : \Gamma \rightarrow \Gamma/(Z \cap \Gamma)$  denotes the natural projection. One can check that

$$\ker(\Theta) \approx \varprojlim_{[\Gamma:H] < \infty} H^1(\pi(H), \mathbb{Z}) \approx H^1(\pi(\Gamma), \mathbb{Q}) \approx \mathbb{Q}^{2n}.$$

Therefore  $\text{Comm}(\Gamma)$  satisfies the short exact sequence

$$1 \rightarrow \mathbb{Q}^{2n} \rightarrow \text{Comm}(\Gamma) \rightarrow \text{GSp}_{2n}(\mathbb{Q}) \rightarrow 1.$$

The action of  $\text{GSp}_{2n}(\mathbb{Q})$  on  $\mathbb{Q}^{2n}$  is the tensor product of the dual representation with the 1-dimensional representation  $\mu : \text{GSp}_{2n}(\mathbb{Q}) \rightarrow \mathbb{Q}^*$  defined by  $\omega(Au, Av) = \mu(A)\omega(u, v)$ .

**4.2. Commensuristic subgroups.** Lattices in simply-connected nilpotent Lie groups provide a source of examples of commensuristic and strongly commensuristic subgroups. Recall that the upper central series  $\gamma^i(G)$  and lower central series  $\gamma_i(G)$  of a group  $G$  are defined inductively as follows. Let  $\gamma^0(G) = 1$ . Suppose that  $\gamma^i(G)$  is a normal subgroup of  $G$ , and let  $\pi : G \rightarrow G/\gamma^i(G)$ . Define  $\gamma^{i+1}(G) = \pi^{-1}(Z(G/\gamma^i(G)))$ . Now let  $\gamma_0(G) = G$ . Supposing  $\gamma_i(G)$  is defined, set  $\gamma_{i+1}(G) = [G, \gamma_i(G)]$ .

**Proposition 4.1.** *Let  $\Gamma \leq N$  be a lattice in a simply-connected nilpotent Lie group. The upper central series of  $\Gamma$  is strongly commensuristic in  $\Gamma$ . The lower central series of  $\Gamma$  is commensuristic.*

*Proof.* A discrete subgroup  $\Delta \leq N$  is a lattice in  $N$  if and only if  $\Delta$  is Zariski-dense in some (equivalently, any) faithful unipotent representation of  $N$  into  $\text{GL}_n(\mathbb{R})$ ; see [Rag72] for a proof. We will show by induction that  $\gamma^k(\Gamma) = \Gamma \cap \gamma^k(N)$  for all  $k$ . The base case is trivial. Now suppose  $\gamma^{i-1}(\Gamma) = \Gamma \cap \gamma^{i-1}(N)$ . Clearly  $\Gamma \cap \gamma^i(N) \subseteq \gamma^i(\Gamma)$ . Suppose  $g \in \gamma^i(\Gamma)$ . By inductive hypothesis,  $\Gamma/\gamma^{i-1}(\Gamma)$  is equal to the image of  $\Gamma$  in  $N/\gamma^{i-1}(N)$ . The image of  $\Gamma$  is Zariski-dense in  $N/\gamma^{i-1}(N)$ . Then  $g$  commutes with every element of  $N/\gamma^{i-1}(N)$  and therefore  $g \in \gamma^i(N)$ .

Now suppose  $\phi : \Gamma_1 \rightarrow \Gamma_2$  is a partial automorphism of  $\Gamma$ . Both  $\Gamma_1$  and  $\Gamma_2$  are lattices in  $N$ , so  $\gamma^k(\Gamma_j) = \Gamma_j \cap \gamma^k(N)$  for  $j = 1, 2$ . It follows that

$$\gamma^k(\Gamma_j) = \Gamma_j \cap \gamma^k(\Gamma) \text{ for } j = 1, 2.$$

Clearly,  $\phi(\gamma^k(\Gamma_1)) = \gamma^k(\Gamma_2)$  for all  $k$ , from which it follows that  $\gamma^k(\Gamma)$  is strongly commensuristic for all  $k$ .

Consider the lower central series  $\gamma_k(\Gamma)$ . Then  $\gamma_k(\Gamma)$  is Zariski-dense in  $\gamma_k(N)$  for all  $k$  by [Bor91, 2.4]. Now suppose  $\phi : \Gamma_1 \rightarrow \Gamma_2$  is a partial automorphism of  $\Gamma$ . Then  $\gamma_k(\Gamma_j)$  is a lattice in  $\gamma_k(N)$  for all  $k$  for  $j = 1, 2$ . Since  $\gamma_k(\Gamma_j) \leq \gamma_k(\Gamma)$ , it follows that  $\gamma_k(\Gamma_j) \leq \gamma_k(\Gamma)$  is of finite index for all  $k$  for  $j = 1, 2$ . Since  $\phi$  clearly takes  $\gamma_k(\Gamma_1)$  to  $\gamma_k(\Gamma_2)$  for all  $k$ , it follows that  $[\phi] \cdot [\gamma_k(\Gamma)] = [\gamma_k(\Gamma)]$  for all  $k$ .  $\square$

**4.3. Commensurations are rational.** Let  $N$  be a simply-connected nilpotent Lie group containing a lattice  $\Gamma$ . Let  $\mathfrak{n}$  denote the Lie algebra of  $N$ . Then  $\mathfrak{n}$  admits rational structure constants by [Rag72, 2.12]. It follows that  $\mathfrak{n}$  admits a basis, unique up to  $\mathbb{Q}$ -defined isomorphism, so that  $\log(\Gamma) \subseteq \mathfrak{n}(\mathbb{Q})$ . Further,  $\text{Aut}(\mathfrak{n})$  is identified with  $\mathbf{A}(\mathbb{R})$  for a  $\mathbb{Q}$ -defined algebraic subgroup  $\mathbf{A} \leq \text{GL}(\mathfrak{n} \otimes \mathbb{C})$  unique up to  $\mathbb{Q}$ -defined isomorphism. It is a standard fact of Lie theory that the exponential map identifies  $\text{Aut}(N)$  with  $\text{Aut}(\mathfrak{n})$ .

This identifies  $\text{Aut}(N)$  with the real points of a  $\mathbb{Q}$ -defined algebraic group  $\mathbf{A}$ . By abuse of notation, we write  $\text{Aut}(N)(\mathbb{Q})$  for the subgroup of  $\text{Aut}(N)$  corresponding to  $\mathbf{A}(\mathbb{Q})$ . The group  $\text{Aut}(N)(\mathbb{Q})$  depends only on  $N$  and  $\Gamma$ .

**Theorem 4.2.** *Let  $\Gamma \leq N$  be a lattice in a simply-connected nilpotent Lie group. Identify  $\text{Aut}(N)$  with the real points of a  $\mathbb{Q}$ -defined algebraic group as above. Then there is an isomorphism*

$$\xi : \text{Comm}(\Gamma) \rightarrow \text{Aut}(N)(\mathbb{Q}).$$

To prove this theorem we will use the fact, due to Malcev, that lattices in nilpotent groups are strongly rigid:

**Theorem 4.3** ([Rag72, 2.11]). *Let  $N_1$  and  $N_2$  be two simply-connected nilpotent Lie groups, with lattices  $\Gamma_1 \leq N_1$  and  $\Gamma_2 \leq N_2$ . Then every isomorphism  $\Gamma_1 \rightarrow \Gamma_2$  extends uniquely to an isomorphism  $N_1 \rightarrow N_2$ .*

*Proof of Theorem 4.2:* Suppose  $\phi : \Gamma_1 \rightarrow \Gamma_2$  is a partial automorphism of  $\Gamma$ . Then  $\phi$  extends to an automorphism  $\Phi \in \text{Aut}(N)$  by Theorem 4.3. Since  $\log(\Gamma)$  is contained in  $\mathfrak{n}(\mathbb{Q})$ , this extension is in  $\text{Aut}(N)(\mathbb{Q})$ . This gives an injective homomorphism

$$\xi : \text{Comm}(\Gamma) \rightarrow \text{Aut}(N)(\mathbb{Q}).$$

Now suppose  $\Phi \in \text{Aut}(N)(\mathbb{Q})$ . It is well-known (for example, see [Rag72, Chapter 2]) that there is a  $\mathbb{Q}$ -defined unipotent algebraic group  $\mathbf{U}$  so that  $N \approx \mathbf{U}(\mathbb{R})$  and  $\Gamma$  is commensurable with  $\mathbf{U}(\mathbb{Z})$ . Then  $\Phi$  extends to a  $\mathbb{Q}$ -defined automorphism of  $\mathbf{U}$ . By [Rag72, 10.4], the group  $\Phi(\Gamma)$  is commensurable with  $\Gamma$ , hence  $\Phi$  induces a commensuration of  $\Gamma$ . It follows that  $\xi$  is surjective.  $\square$

## 5. THE ALGEBRAIC HULL OF A POLYCYCLIC GROUP

**5.1. Polycyclic groups.** We briefly review the general theory of lattices in solvable Lie groups. See [Rag72] for a general reference, and [Seg83] for the theory of polycyclic groups.

**Definition 5.1.** A group  $\Gamma$  is *polycyclic* if there is a subnormal series

$$(1) \quad 1 \triangleleft \Gamma_1 \triangleleft \Gamma_2 \triangleleft \cdots \triangleleft \Gamma$$

so that  $\Gamma_i/\Gamma_{i-1}$  is cyclic for each  $i$ .

The *Hirsch number* of  $\Gamma$ , denoted  $\text{rank}(\Gamma)$ , is the number of  $i$  such that  $\Gamma_i/\Gamma_{i-1}$  is infinite cyclic. Hirsch number is independent of choice of such subnormal series, and is invariant under passage to finite index subgroups. Every polycyclic group contains a finite index subgroup admitting a subnormal series (1) such that each  $\Gamma_i/\Gamma_{i-1}$  is infinite cyclic. Call such a group *strongly polycyclic*. It is a theorem of Wang that every lattice in a simply-connected solvable Lie group is strongly polycyclic.

Every polycyclic group  $\Gamma$  admits a unique maximal normal nilpotent subgroup, called the *Fitting subgroup*, denoted  $\text{Fitt}(\Gamma)$ . If  $\Gamma$  is a strongly polycyclic group, then  $\text{Fitt}(\Gamma)$  is isomorphic to a lattice in a simply-connected nilpotent Lie group  $N$ . By Theorem 4.3, conjugation extends to a representation  $\tilde{\sigma} : \Gamma \rightarrow \text{Aut}(N)$ . If  $\mathfrak{n}$  is the Lie algebra of  $N$ , then by identifying  $\text{Aut}(N)$  with  $\text{Aut}(\mathfrak{n}) \subseteq \text{GL}(\mathfrak{n})$  we have a representation  $\sigma : \Gamma \rightarrow \text{GL}(\mathfrak{n})$ .

**Proposition 5.2** ([Rag72, 4.10]). *Let  $\Gamma$  be strongly polycyclic, and  $\sigma : \Gamma \rightarrow \text{GL}(\mathfrak{n})$  as above. Then*

$$\text{Fitt}(\Gamma) = \{\gamma \in \Gamma \mid \sigma(\gamma) \text{ is unipotent}\}.$$

**Lemma 5.3.** *Let  $\Gamma$  be a strongly polycyclic group, with  $\Gamma_1 \leq \Gamma$  a finite index subgroup. Then  $\text{Fitt}(\Gamma_1) = \text{Fitt}(\Gamma) \cap \Gamma_1$ .*

*Proof.* It is clear that  $\text{Fitt}(\Gamma_1) \cap \Gamma \leq \text{Fitt}(\Gamma)$ , so we have only to show that  $\text{Fitt}(\Gamma_1) \leq \text{Fitt}(\Gamma)$ . Let  $N$  be the Lie group containing  $\text{Fitt}(\Gamma)$  as a lattice, and  $N_1$  the Lie group containing  $\text{Fitt}(\Gamma_1)$  as a lattice. Then  $\Gamma_1 \cap \text{Fitt}(\Gamma)$  is a lattice in  $N$ . It follows that the inclusion  $\Gamma_1 \cap \text{Fitt}(\Gamma) \rightarrow \text{Fitt}(\Gamma_1)$  extends to an embedding of Lie groups  $i : N \rightarrow N_1$  by [Rag72, 2.11]. This gives an embedding of  $\mathfrak{n}$  as a Lie subalgebra of  $\mathfrak{n}_1$ . Let  $\sigma : \Gamma \rightarrow \text{GL}(\mathfrak{n})$  and  $\sigma_1 : \Gamma_1 \rightarrow \text{GL}(\mathfrak{n}_1)$  be as above. Then  $\mathfrak{n}$  is invariant under  $\sigma_1(\Gamma_1)$  because  $\text{Fitt}(\Gamma)$  is normal in  $\Gamma$ . Suppose  $\gamma \in \text{Fitt}(\Gamma_1)$ . Then by Proposition 5.2,  $\sigma_1(\gamma)$  is unipotent. It follows that  $\sigma(\gamma)$  is unipotent, and so  $\gamma \in \text{Fitt}(\Gamma)$  by Proposition 5.2.  $\square$

**Corollary 5.4.** *If  $\Gamma$  is strongly polycyclic then  $\text{Fitt}(\Gamma)$  is strongly commensurist.*  $\square$

**5.2. Algebraic hulls.** Our main tool for understanding commensurations of a polycyclic group  $\Gamma$  will be its algebraic hull. Roughly speaking, the algebraic hull is the largest algebraic group in which  $\Gamma$  is Zariski-dense. The fact that it is the “largest” is important for the extension of commensurations to algebraic automorphisms. The original construction is due to Mostow [Mos70], with an alternate construction in [Rag72]. More recently, algebraic hulls have been constructed for certain virtually polycyclic groups by Baues in [Bau04]. We will need only the classical algebraic hull.

**Definition 5.5** (Algebraic hull). Suppose  $\Gamma$  is a strongly polycyclic group. An *algebraic hull* of  $\Gamma$  is a  $\mathbb{Q}$ -defined algebraic group  $\mathbf{H}$  with an embedding  $i : \Gamma \rightarrow \mathbf{H}(\mathbb{Q})$  so that

- (H1)  $i(\Gamma)$  is Zariski-dense in  $\mathbf{H}$ ,
- (H2)  $Z_{\mathbf{H}}(\mathbf{U}_{\mathbf{H}}) \leq \mathbf{U}_{\mathbf{H}}$ , where  $\mathbf{U}_{\mathbf{H}}$  is the unipotent radical of  $\mathbf{H}$ ,
- (H3)  $\dim(\mathbf{U}_{\mathbf{H}}) = \text{rank}(\Gamma)$ , and
- (H4)  $i(\Gamma) \cap \mathbf{H}(\mathbb{Z})$  is of finite index in  $i(\Gamma)$ .

Algebraic hulls exist for all strongly polycyclic groups. See [Rag72] for a construction, which uses the fact, due to Auslander, that any polycyclic group has a faithful embedding to  $\text{GL}_n(\mathbb{Z})$  for some  $\mathbb{Z}$ . The importance of the algebraic hull is its uniqueness:

**Lemma 5.6** ([Rag72, 4.41]). *Suppose  $\Gamma_1$  and  $\Gamma_2$  are two strongly polycyclic groups, and  $\phi : \Gamma_1 \rightarrow \Gamma_2$  is an isomorphism. Let  $i_1 : \Gamma_1 \rightarrow \mathbf{H}_1$  and  $i_2 : \Gamma_2 \rightarrow \mathbf{H}_2$  be algebraic hulls for  $\Gamma_1$  and  $\Gamma_2$ , respectively. Then  $\phi$  extends to a  $\mathbb{Q}$ -defined isomorphism  $\Phi : \mathbf{H}_1 \rightarrow \mathbf{H}_2$ .*

**Example 5.7.** The condition (H3) is necessary for the extension of automorphisms. Consider the following. Let  $\psi \in \text{Aut}(\mathbb{Z}^2)$  be the automorphism given by  $\psi(1, 0) = (2, 1)$  and  $\psi(0, 1) = (1, 1)$ . Then the group  $\Gamma = \mathbb{Z} \times (\mathbb{Z}^2 \rtimes \langle \psi \rangle)$  has an obvious embedding as a lattice in the solvable Lie group  $G = \mathbb{R} \times (\mathbb{R}^2 \rtimes \mathbb{R})$ , where the latter  $\mathbb{R}$  factor acts on  $\mathbb{R}^2$  by a 1-parameter subgroup  $\psi_t$  of  $\text{GL}_2(\mathbb{R})$  with  $\psi_1 = \psi$ . Parametrize  $G$  by writing elements  $(x, y, z, t)$  for  $x, y, z, t \in \mathbb{R}$ , and let  $M_t$  be the matrix in  $\text{GL}_2(\mathbb{R})$  corresponding to  $\psi_t$  in these coordinates. There is a faithful matrix representation of  $G$  in  $\text{GL}_4(\mathbb{R})$  taking  $\Gamma$  into  $\text{GL}_4(\mathbb{Z})$ :

$$\rho_1 : (x, y, z, t) \mapsto \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & & & y \\ 0 & M_t & & z \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The Zariski-closure  $\mathbf{A}$  of  $\rho_1(\Gamma)$  in  $\text{GL}_4(\mathbb{C})$  has the structure  $\mathbf{A} = \mathbf{U} \rtimes \mathbf{T}$ , where  $\mathbf{T} \approx \mathbb{C}^*$  is a 1-dimensional torus and  $\mathbf{U} \approx \mathbb{C}^3$ . Note that  $\mathbf{A}$  is a  $\mathbb{Q}$ -defined algebraic group satisfying (H1), (H2), and (H4). However,  $\dim(\mathbf{U}_{\mathbf{A}}) = 3$  while  $\text{rank}(\Gamma) = 4$ , so  $\mathbf{A}$  is not an algebraic hull of  $\Gamma$ .

The function  $\phi : \Gamma \rightarrow \Gamma$  defined by  $\phi(x, y, z, t) = (x + t, y, z, t)$  defines an automorphism of  $\Gamma$  that cannot be extended to an algebraic automorphism of  $\mathbf{A}$ . If  $\phi$  were to extend to  $\mathbf{A}$ , then the extension would take

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The former is semisimple, while the latter is not. Any algebraic automorphism must take semisimple elements to semisimple elements, so this cannot extend.

The algebraic hull  $\mathbf{H}$  of  $\Gamma$  may be realized as the Zariski-closure of the following embedding  $\Gamma \rightarrow \text{GL}_6(\mathbb{Z})$ :

$$\rho_2 : (x, y, z, t) \mapsto \begin{pmatrix} 1 & 0 & 0 & x & & \\ 0 & & & y & & \\ 0 & M_t & & z & & \\ 0 & 0 & 0 & 1 & & \\ & & & & 1 & t \\ & & & & 0 & 1 \end{pmatrix}.$$

The Zariski-closure of  $\rho_2(\Gamma)$  in  $\text{GL}_6(\mathbb{C})$  is isomorphic to  $\mathbb{C}^2 \times (\mathbb{C}^2 \rtimes \mathbb{C}^*)$ , since it includes the closures of both the semisimple and unipotent parts of

$\rho_2(0, 0, 0, t)$  for all  $t \in \mathbb{Z}$ . Then the automorphism  $\phi$  extends to the algebraic automorphism of  $\mathbf{H}$

$$\begin{pmatrix} 1 & 0 & 0 & x \\ 0 & & & y \\ 0 & M_* & & z \\ 0 & 0 & 0 & 1 \\ & & & 1 & t \\ & & & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & x+t \\ 0 & & & y \\ 0 & M_* & & z \\ 0 & 0 & 0 & 1 \\ & & & 1 & t \\ & & & 0 & 1 \end{pmatrix},$$

where  $M_*$  is a matrix belonging to the torus subgroup  $\mathbb{C}^* \leq \mathbf{H}$ .  $\square$

We wish to use rigidity of the algebraic hull to construct an embedding of  $\text{Comm}(\Gamma)$  into  $\text{Aut}_{\mathbb{Q}}(\mathbf{H})$  analogous to the use of Malcev rigidity in Theorem 4.2. For this, the natural setting is the Zariski-connected component of the identity of the algebraic hull.

**Definition 5.8** (Virtual algebraic hull). Let  $\Gamma$  be a virtually polycyclic group. A  $\mathbb{Q}$ -defined algebraic group  $\mathbf{H}$  is a *virtual algebraic hull* of  $\Gamma$  if

- (1)  $\mathbf{H}$  is connected, and
- (2) there is a finite index strongly polycyclic subgroup  $\Delta \leq \Gamma$  such that  $\mathbf{H}$  is an algebraic hull of  $\Delta$ .

**Lemma 5.9.** *Every virtually polycyclic group has a virtual algebraic hull.*

*Proof.* Suppose  $\Gamma$  is virtually polycyclic. Let  $\tilde{\Gamma} \leq \Gamma$  be any finite index strongly polycyclic subgroup. Let  $\tilde{\mathbf{H}}$  be an algebraic hull for  $\tilde{\Gamma}$ . Then the identity component  $\tilde{\mathbf{H}}^0$  is of finite index in  $\tilde{\mathbf{H}}$ . Let  $\Delta = \tilde{\Gamma} \cap \tilde{\mathbf{H}}^0$  and  $\mathbf{H} = \tilde{\mathbf{H}}^0$ .

Clearly  $\Delta \leq \tilde{\Gamma}$  is a finite index strongly polycyclic subgroup. The closure of  $\Delta$  is contained in  $\mathbf{H}$ . Conversely, the closure of  $\Delta$  is of finite index in  $\tilde{\mathbf{H}}$ , and so must contain  $\mathbf{H}$ . Therefore (H1) of Definition 5.5 is satisfied. The unipotent radical of  $\mathbf{H}$  is equal to the unipotent radical of  $\tilde{\mathbf{H}}$ , so (H2) is satisfied. We know  $\text{rank}(\Delta) = \text{rank}(\tilde{\Gamma})$ , so (H3) holds. And clearly (H4) holds for  $\Delta$  since it does for  $\tilde{\Gamma}$ . So  $\mathbf{H}$  is an algebraic hull of  $\Delta$ .  $\square$

**Definition 5.10** (Fitting subgroup). Suppose  $\mathbf{H}$  is the virtual algebraic hull of a virtually polycyclic group  $\Gamma$ , and let  $\Delta \leq \Gamma$  be a finite index strongly polycyclic group with an embedding in  $\mathbf{H}$  satisfying (H1)–(H4). Define  $\text{Fitt}(\mathbf{H})$ , the *Fitting subgroup* of  $\mathbf{H}$ , to be the Zariski-closure of  $\text{Fitt}(\Delta)$  in  $\mathbf{H}$ .

The Fitting subgroup is independent of choice of  $\Delta$  by Lemma 5.3. Note that  $[\mathbf{H}, \mathbf{H}] \leq \text{Fitt}(\mathbf{H})$ , by [BG06, 4.6].

**Lemma 5.11.** *Let  $\Gamma$  be virtually polycyclic. The virtual algebraic hull of  $\Gamma$  is unique up to  $\mathbb{Q}$ -defined isomorphism.*

*Proof.* Suppose  $\mathbf{H}_1$  and  $\mathbf{H}_2$  are two virtual algebraic hulls of  $\Gamma$ . Let  $\Gamma_1$  and  $\Gamma_2$  be two finite index strongly polycyclic subgroups of  $\Delta$  with injections  $i_1 : \Gamma_1 \rightarrow \mathbf{H}_1(\mathbb{Q})$  and  $i_2 : \Gamma_2 \rightarrow \mathbf{H}_2(\mathbb{Q})$  satisfying (H1)–(H4). Then  $\Gamma_1 \cap \Gamma_2$



is of finite index in both  $\Gamma_1$  and  $\Gamma_2$ . Because  $\mathbf{H}_1$  and  $\mathbf{H}_2$  are connected, both  $i_1|_{\Gamma_1 \cap \Gamma_2}$  and  $i_2|_{\Gamma_1 \cap \Gamma_2}$  satisfy (H1)–(H4) for  $\Gamma_1 \cap \Gamma_2$  in place of  $\Gamma$ . It follows from Lemma 5.6 that there is a  $\mathbb{Q}$ -defined isomorphism  $\Phi : \mathbf{H}_1 \rightarrow \mathbf{H}_2$  extending  $i_2 \circ i_1|_{\Gamma_1 \cap \Gamma_2}^{-1}$ .  $\square$

**Corollary 5.12.** *Suppose  $\Gamma$  is virtually polycyclic with virtual algebraic hull  $\mathbf{H}$ . There is an embedding*

$$(2) \quad \xi : \text{Comm}(\Gamma) \rightarrow \text{Aut}_{\mathbb{Q}}(\mathbf{H}).$$

*Proof.* Let  $\Delta \leq \Gamma$  be a finite index strongly polycyclic subgroup with an embedding  $i : \Delta \rightarrow \mathbf{H}(\mathbb{Q})$  satisfying (H1)–(H4). Suppose  $\phi : \Delta_1 \rightarrow \Delta_2$  is a partial automorphism of  $\Delta$ . Then  $\mathbf{H}$  is an algebraic hull for both  $\Delta_1$  and  $\Delta_2$  by connectedness, so  $\phi$  extends to  $\Phi \in \text{Aut}_{\mathbb{Q}}(\mathbf{H})$ . Equivalent partial automorphisms clearly give rise to equal extensions. The assignment  $\phi \mapsto \Phi$  gives an injective homomorphism  $\text{Comm}(\Delta) \rightarrow \text{Aut}_{\mathbb{Q}}(\mathbf{H})$  by density of  $\Delta_1$  and  $\Delta_2$ . The proof is complete since  $\text{Comm}(\Gamma) \approx \text{Comm}(\Delta)$ .  $\square$

Though these results are stated generally for virtually polycyclic groups, the reduction to the identity component is necessary even for lattices in simply-connected solvable Lie groups.

**Example 5.13.** Let  $G = \mathbb{R}^2 \rtimes \widetilde{SO(2)}$ . Then  $G$  is homeomorphic to  $\mathbb{R}^3$ . Parametrize  $G$  in this way, so that conjugation by  $(0, 0, t)$  acts as rotation by angle  $t$  in the  $t = 0$  plane. Let  $\Gamma$  be the lattice

$$\Gamma = \{(x, y, t) \in G \mid x, y \in \mathbb{Z} \text{ and } t \in \pi\mathbb{Z}\}.$$

Then  $\Gamma$  contains a subgroup of index 2 isomorphic to  $\mathbb{Z}^3$ , so its virtual algebraic hull is a unipotent group isomorphic to  $\mathbb{C}^3$ . The algebraic hull of  $\Gamma$  is a (disconnected) index 2 extension of  $\mathbb{C}^3$ .  $\square$

There is an analogous construction of algebraic hulls for simply-connected solvable Lie groups  $G$ , though they are only  $\mathbb{R}$ -defined rather than  $\mathbb{Q}$ -defined.

**Definition 5.14** (Algebraic hull). Suppose  $G$  is a simply-connected solvable Lie group. An *algebraic hull* of  $G$  is an  $\mathbb{R}$ -defined algebraic group  $\mathbf{H}$  with an embedding  $i : G \rightarrow \mathbf{H}(\mathbb{R})$  so that

- (1)  $i(G)$  is Zariski-dense in  $\mathbf{H}$ ,
- (2)  $Z_{\mathbf{H}}(\mathbf{U}_{\mathbf{H}}) \leq \mathbf{U}_{\mathbf{H}}$ , where  $\mathbf{U}_{\mathbf{H}}$  is the unipotent radical of  $\mathbf{H}$ , and
- (3)  $\dim(\mathbf{U}_{\mathbf{H}}) = \dim(G)$ .

The algebraic hull of the group  $G$  in Example 5.13 contains a 1-dimensional torus, hence is strictly larger than the algebraic hull of  $\Gamma$ . See [BK11] for a detailed discussion of the relationship between the algebraic hull of a lattice and the algebraic hull of the ambient Lie group. We use this theory in §8.

**5.3. Unipotent shadow.** Much of the theory of lattices in solvable Lie groups builds on the much easier theory of lattices in nilpotent Lie group. A common tool is the *unipotent shadow*. The following proposition summarizes the theory of unipotent shadows of strongly polycyclic groups in algebraic hulls as explained in Sections 5 and 7 of [BG06]. For the reader's convenience, we include a sketch of a proof.

**Proposition 5.15** ([BG06]). *Suppose  $\Gamma$  is a virtually polycyclic group with virtual algebraic hull  $\mathbf{H}$ . Let  $\mathbf{F}$  be the Fitting subgroup of  $\mathbf{H}$ . There is a strongly polycyclic subgroup  $\Lambda \leq \mathbf{H}(\mathbb{Q})$  abstractly commensurable with  $\Gamma$  so that:*

- (1) *There is a nilpotent subgroup  $C \leq \Lambda$  so that  $\Lambda = \text{Fitt}(\Lambda) \cdot C$ .*
- (2) *There is a  $\mathbb{Q}$ -defined maximal torus  $\mathbf{T} \leq \mathbf{H}$  with centralizer  $\mathbf{D} \leq \mathbf{H}$  so that  $C = \Lambda \cap \mathbf{D}$ , and  $C$  is Zariski-dense in  $\mathbf{D}$ .*
- (3) *The subgroup  $\theta \leq \mathbf{U}_{\mathbf{H}}(\mathbb{Q})$  generated by  $\text{Fitt}(\Lambda)$  and  $C_u$  is a finitely generated subgroup Zariski-dense in  $\mathbf{U}_{\mathbf{H}}$ , such that  $\text{Fitt}(\Lambda) = \theta \cap \mathbf{F}$ .*

*Sketch of proof:* Let  $\Delta$  be a strongly polycyclic subgroup of  $\Gamma$  so that  $\mathbf{H}$  is an algebraic hull of  $\Delta$ . Fix any maximal  $\mathbb{Q}$ -defined torus  $\mathbf{T} \leq \mathbf{H}$ , and let  $\mathbf{D}$  be the normalizer of  $\mathbf{T}$  in  $\mathbf{H}$ . Then  $\mathbf{D}$  is a connected nilpotent  $\mathbb{Q}$ -defined subgroup of  $\mathbf{H}$  that centralizes  $\mathbf{T}$ . By replacing  $\text{Fitt}(\Delta)$  with a finite index supergroup, we obtain a strongly polycyclic group  $\Lambda \leq \mathbf{H}(\mathbb{Q})$  commensurable with  $\Delta$  for which the group  $C = \Lambda \cap \mathbf{D}$  is Zariski-dense in  $\mathbf{D}$  and satisfies  $\Lambda = \text{Fitt}(\Lambda) \cdot C$ . The group  $\Lambda$  is called a *thickening* of  $\Delta$ , and  $C$  is called a *nilpotent supplement* in  $\Lambda$ .

We now want to construct the group  $\theta$  by taking the unipotent parts of elements of  $\Lambda$ . For every  $c \in \mathbf{D}$ , let  $c_s$  and  $c_u$  denote the semisimple and unipotent parts, respectively, of its Jordan decomposition in  $\mathbf{D}$ . Because  $\mathbf{D}$  centralizes  $\mathbf{T}$ , the map  $c \mapsto c_u$  is a homomorphism  $\mathbf{D} \rightarrow \mathbf{U}_{\mathbf{H}}$ . Define  $\theta$  to be the subgroup of  $\mathbf{U}_{\mathbf{H}}(\mathbb{Q})$  generated by  $\text{Fitt}(\Lambda)$  and  $C_u$ . By replacing  $\Lambda$  with a further thickening, we can guarantee that  $\theta \cap \mathbf{F} = \text{Fitt}(\Lambda)$ . Such a group  $\theta$  is called a *good unipotent shadow*.  $\square$

**5.4. Algebraic structure of  $\text{Aut}(\mathbf{H})$ .** Suppose  $\Gamma \leq G$  is a lattice in a simply-connected solvable Lie group, and let  $\mathbf{H}$  be its virtual algebraic hull. We recall the structure of  $\text{Aut}_{\mathbb{Q}}(\mathbf{H})$  explained in Section 3 of [BG06]. Let  $\mathbf{U}$  be the unipotent radical of  $\mathbf{H}$ . Fix a  $\mathbb{Q}$ -defined maximal torus  $\mathbf{T} \leq \mathbf{H}$ . There is a decomposition  $\mathbf{H} = \mathbf{U} \rtimes \mathbf{T}$ . Define

$$(3) \quad \text{Aut}(\mathbf{H})_{\mathbf{T}} = \{\Phi \in \text{Aut}(\mathbf{H}) \mid \Phi(\mathbf{T}) = \mathbf{T}\}.$$

By property (H2) of the algebraic hull, the restriction map  $\text{Aut}(\mathbf{H})_{\mathbf{T}} \rightarrow \text{Aut}(\mathbf{U})$  is injective. Its image is a  $\mathbb{Q}$ -defined closed subgroup of  $\text{Aut}(\mathbf{U})$ . The map

$$(4) \quad \begin{aligned} \Theta : \mathbf{U} \rtimes \text{Aut}(\mathbf{H})_{\mathbf{T}} &\rightarrow \text{Aut}(\mathbf{H}) \\ (u, \Phi) &\mapsto \text{Inn}_u \circ \Phi \end{aligned}$$

is a surjection with  $\mathbb{Q}$ -defined kernel. The quotient  $\mathbf{U} \rtimes \text{Aut}(\mathbf{H})_{\mathbf{T}} / \ker(\Theta)$  is a  $\mathbb{Q}$ -defined algebraic group, which gives  $\text{Aut}(\mathbf{H})$  the structure of a  $\mathbb{Q}$ -defined algebraic group. Because  $\ker(\Theta)$  is unipotent, it follows from the discussion of [PR94, 2.2.3] (see also [BG06, 3.6]) that there is a group isomorphism

$$(5) \quad \text{Aut}_{\mathbb{Q}}(\mathbf{H}) \approx \mathbf{U}(\mathbb{Q}) \rtimes \text{Aut}(\mathbf{H})_{\mathbf{T}}(\mathbb{Q}) / (\ker \Theta)(\mathbb{Q}).$$

Thus the algebraic structure of  $\text{Aut}(\mathbf{H})$  is such that  $\text{Aut}_{\mathbb{Q}}(\mathbf{H}) = \text{Aut}(\mathbf{H})(\mathbb{Q})$ .

**5.5. A finite index subgroup of  $\text{Comm}(\Gamma)$ .** Let  $\Gamma$ ,  $\mathbf{H}$ , and  $\mathbf{U}$  be as above. Let  $\mathbf{F} = \text{Fitt}(\mathbf{H})$ . Define

$$(6) \quad \mathcal{A}_{\mathbf{H}|\mathbf{U}} = \left\{ \Phi \in \text{Aut}(\mathbf{H}) \mid \Phi|_{\mathbf{H}/\mathbf{U}} = \text{Id}_{\mathbf{H}/\mathbf{U}} \right\}.$$

**Lemma 5.16.** *The subgroup  $\mathcal{A}_{\mathbf{H}|\mathbf{U}} \leq \text{Aut}(\mathbf{H})$  is of finite index.*

*Proof.* The quotient  $\mathbf{H}/\mathbf{U}$  is a  $\mathbb{Q}$ -defined torus. By Lemma 2.5, the identity component  $\text{Aut}(\mathbf{H})^0$  acts trivially on the torus  $\mathbf{H}/\mathbf{U}$ , and so  $\text{Aut}(\mathbf{H})^0 \leq \mathcal{A}_{\mathbf{H}|\mathbf{U}}$ . The claim follows since  $[\text{Aut}(\mathbf{H})^0 : \text{Aut}(\mathbf{H})] < \infty$ .  $\square$

Let  $N_{\text{Aut}(\mathbf{H})}(\mathbf{F})$  denote the subgroup of  $\text{Aut}(\mathbf{H})$  preserving  $\mathbf{F}$ . Define

$$(7) \quad \mathcal{A}_{\mathbf{H}|\mathbf{F}} = \left\{ \Phi \in N_{\text{Aut}(\mathbf{H})}(\mathbf{F}) \mid \Phi|_{\mathbf{H}/\mathbf{F}} = \text{Id}_{\mathbf{H}/\mathbf{F}} \right\}.$$

By Corollary 5.4, the image of the map  $\xi : \text{Comm}(\Gamma) \rightarrow \text{Aut}(\mathbf{H})$  preserves  $\mathbf{F}$ . Define

$$(8) \quad \text{Comm}_{\mathbf{H}|\mathbf{F}}(\Gamma) = \xi^{-1}(\mathcal{A}_{\mathbf{H}|\mathbf{F}}).$$

**Lemma 5.17.**  $[\text{Comm}(\Gamma) : \text{Comm}_{\mathbf{H}|\mathbf{F}}(\Gamma)] < \infty$ .

*Proof.* By Lemma 5.16, it suffices to show that  $\text{Comm}_{\mathbf{H}|\mathbf{F}}(\Gamma) = \text{Comm}(\Gamma) \cap \mathcal{A}_{\mathbf{H}|\mathbf{U}}$ . Since  $\mathcal{A}_{\mathbf{H}|\mathbf{F}} \leq \mathcal{A}_{\mathbf{H}|\mathbf{U}}$ , it is clear that  $\text{Comm}_{\mathbf{H}|\mathbf{F}}(\Gamma) \leq \text{Comm}(\Gamma) \cap \mathcal{A}_{\mathbf{H}|\mathbf{U}}$ . On the other hand, suppose that  $[\phi] \in \text{Comm}(\Gamma) \cap \mathcal{A}_{\mathbf{H}|\mathbf{U}}$ . Without loss of generality, assume that  $\phi$  is a partial automorphism of a finite index subgroup  $\Delta \leq \Gamma$  for which  $\mathbf{H}$  is an algebraic hull. By Proposition 5.2, we have that  $\Delta \cap \mathbf{U} = \text{Fitt}(\Delta)$ . It follows that if  $\phi(\gamma)\gamma^{-1} \in \mathbf{U}$  for some  $\gamma \in \Delta$ , then  $\phi(\gamma)\gamma^{-1} \in \text{Fitt}(\Delta)$ . Therefore  $[\phi] \in \text{Comm}_{\mathbf{H}|\mathbf{F}}(\Gamma)$ .  $\square$

The structure of  $\mathcal{A}_{\mathbf{H}|\mathbf{F}}$  can be made more explicit, following Section 3.3 of [BG06]. Let  $\mathbf{T}$  denote a maximal  $\mathbb{Q}$ -defined torus in  $\mathbf{H}$ . Define

$$(9) \quad \mathcal{A}_{\mathbf{T}}^1 = \left\{ \Phi \in \mathcal{A}_{\mathbf{H}|\mathbf{F}} \mid \Phi(\mathbf{T}) = \mathbf{T}, \Phi|_{\mathbf{T}} = \text{id}_{\mathbf{T}} \right\},$$

$$(10) \quad \text{Inn}_{\mathbf{F}}^{\mathbf{H}} = \left\{ \Phi \in \text{Aut}(\mathbf{H}) \mid \Phi(x) = fxf^{-1} \text{ for some } f \in \mathbf{F} \right\}.$$

Clearly  $\text{Inn}_{\mathbf{F}}^{\mathbf{H}}$  and  $\mathcal{A}_{\mathbf{T}}^1$  are both  $\mathbb{Q}$ -defined subgroups of  $\mathcal{A}_{\mathbf{H}|\mathbf{F}}$ , and  $\text{Inn}_{\mathbf{F}}^{\mathbf{H}}$  is normal in  $\mathcal{A}_{\mathbf{H}|\mathbf{F}}$ . Let  $(\mathcal{A}_{\mathbf{H}|\mathbf{F}})_{\mathbb{Q}}$  denote the group of  $\mathbb{Q}$ -defined automorphisms in  $\mathcal{A}_{\mathbf{H}|\mathbf{F}}$ . Because any two maximal  $\mathbb{Q}$ -defined tori are conjugate by an element of  $[\mathbf{H}, \mathbf{H}](\mathbb{Q}) \leq \mathbf{F}(\mathbb{Q})$ , we have

**Lemma 5.18.**  $\mathcal{A}_{\mathbf{H}|\mathbf{F}} = \text{Inn}_{\mathbf{F}}^{\mathbf{H}} \cdot \mathcal{A}_{\mathbf{T}}^1$ . Moreover,  $(\mathcal{A}_{\mathbf{H}|\mathbf{F}})_{\mathbb{Q}} = \text{Inn}_{\mathbf{F}}^{\mathbf{H}}(\mathbb{Q}) \cdot \mathcal{A}_{\mathbf{T}}^1(\mathbb{Q})$ .

*Proof.* See [BG06, 3.13]. The latter statement follows from equation (5); cf. [BG06, 3.6].  $\square$

## 6. COMMENSURATIONS OF LATTICES IN SOLVABLE GROUPS

**6.1. Example: Sol lattice.** Let  $\psi : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$  be the automorphism defined by  $\psi(1, 0) = (2, 1)$  and  $\psi(0, 1) = (1, 1)$ . Let  $C$  be the infinite cyclic group generated by  $\psi$ , and define  $\Gamma = \mathbb{Z}^2 \rtimes C$ . Note that  $\Gamma$  is a lattice in 3-dimensional SOL geometry. We have that  $\text{Fitt}(\Gamma) = \mathbb{Z}^2$ , so there are induced maps

$$r : \text{Comm}(\Gamma) \rightarrow \text{Comm}(\mathbb{Z}^2) \approx \text{GL}_2(\mathbb{Q})$$

and

$$\pi : \text{Comm}(\Gamma) \rightarrow \text{Comm}(C) \approx \mathbb{Q}^*.$$

Suppose  $\phi : \Gamma_1 \rightarrow \Gamma_2$  is a partial automorphism of  $\Gamma$ . There are nonzero  $p, q$  so that  $\pi(\phi)[\psi^q] = [\psi^p]$ . Using the fact that  $\phi$  is an isomorphism, we have

$$(11) \quad \phi(\psi^q(v)) = \psi^p(\phi(v))$$

for all  $v \in \Gamma_1 \cap \mathbb{Z}^2$ . Since  $\Gamma_1 \cap \mathbb{Z}^2$  spans  $\mathbb{Z}^2 \otimes \mathbb{Q}$ , it follows that  $r(\phi)$  conjugates  $\psi^q$  to  $\psi^p$  in  $\text{GL}_2(\mathbb{Q})$ . Therefore  $p = \pm q$  since  $\psi$  has an eigenvalue not on the unit circle. It follows that there is an index 2 subgroup  $\text{Comm}^+(\Gamma)$  so that  $\pi$  is trivial when restricted to  $\text{Comm}^+(\Gamma)$ .

Let  $Z_{\text{GL}_2(\mathbb{Q})}(\psi)$  denote the centralizer of  $\psi$  in  $\text{GL}_2(\mathbb{Q})$ , which is equal to the set of matrices in the 1-parameter subgroup of  $\text{GL}_2(\mathbb{R})$  through  $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$  with rational entries. From (11) we see that  $r(\phi) \in Z_{\text{GL}_2(\mathbb{Q})}(\psi)$  for all  $\phi \in \text{Comm}^+(\Gamma)$ . Moreover, it is clear that the induced map  $\bar{r} : \text{Comm}^+(\Gamma) \rightarrow Z_{\text{GL}_2(\mathbb{Q})}(\psi)$  is surjective. Let  $K = \ker(\bar{r})$ . Every  $\phi \in K$  is of the form  $\phi(v, \psi^p) = (v + \rho(\psi^p), \psi^p)$  for a cocycle  $\rho : H \rightarrow \mathbb{Z}^2$  defined on some finite index subgroup  $H \leq C$ . One can show that

$$K = \varprojlim_{[C:H] < \infty} H^1(H, \mathbb{Z}^2) \approx H^1(C, \mathbb{Q}^2) \approx \mathbb{Q}^2.$$

Therefore  $\text{Comm}^+(\Gamma)$  satisfies the short exact sequence

$$1 \rightarrow \mathbb{Q}^2 \rightarrow \text{Comm}^+(\Gamma) \rightarrow Z_{\text{GL}_2(\mathbb{Q})}(\psi) \rightarrow 1.$$

This sequence splits, and the action of  $Z_{\text{GL}_2(\mathbb{Q})}(\psi)$  on  $\mathbb{Q}^2$  is the standard action.

**6.2. Commensurations of solvable lattices are rational.** We continue to use the notation developed in §5. Given a lattice  $\Gamma$  in a connected, simply-connected solvable Lie group, let  $\mathbf{H}$  denote its virtual algebraic hull with Fitting subgroup  $\mathbf{F}$  and  $\mathbb{Q}$ -defined maximal torus  $\mathbf{T}$ . Then  $\mathcal{A}_{\mathbf{H}/\mathbf{F}}$  denotes the group of automorphisms of  $\mathbf{H}$  preserving  $\mathbf{F}$  and trivial on  $\mathbf{H}/\mathbf{F}$ . Let  $\text{Inn}_{\mathbf{F}}^{\mathbf{H}}$  denote the group of automorphisms of  $\mathbf{H}$  induced by conjugation by elements of  $\mathbf{F}$ , and  $\mathcal{A}_{\mathbf{T}}^1$  denote the group of automorphisms fixing  $\mathbf{T}$ .

**Theorem 6.1.** *Let  $\Gamma$  be a lattice in a connected, simply-connected solvable Lie group. Let  $\mathbf{H}$  be the virtual algebraic hull of  $\Gamma$ , with  $\mathbf{F} = \text{Fitt}(\mathbf{H})$ . The map  $\xi : \text{Comm}(\Gamma) \rightarrow \text{Aut}(\mathbf{H})$  induces an isomorphism of groups*

$$\text{Comm}_{\mathbf{H}|\mathbf{F}}(\Gamma) \approx (\mathcal{A}_{\mathbf{H}|\mathbf{F}})_{\mathbb{Q}}.$$

The proof of the theorem is in two steps. First we show that  $\text{Inn}_{\mathbf{F}}^{\mathbf{H}}(\mathbb{Q}) \leq \xi(\text{Comm}(\Gamma))$ , and second that  $\mathcal{A}_{\mathbf{T}}^1(\mathbb{Q}) \leq \xi(\text{Comm}(\Gamma))$ . The unipotent shadow will be our main tool. Before we start the proof of Theorem 6.1, we prove an important technical lemma that will be used again in §8.

**Lemma 6.2.** *Let  $\mathbf{U}$  be a  $\mathbb{Q}$ -defined unipotent algebraic group and  $\theta' \leq \mathbf{U}(\mathbb{Q})$  be a finitely generated, Zariski-dense subgroup. Let  $P$  be a group acting on  $\mathbf{U}$  by algebraic group automorphisms preserving  $\theta'$ . Suppose  $f \in \mathbf{U}(\mathbb{Q})$ . There is some finite index subgroup  $P'' \leq P$  so that  $f(p \cdot f^{-1}) \in \theta'$  for all  $p \in P''$ .*

*Proof.* Let  $\mathfrak{u}$  be the Lie algebra of  $\mathbf{U}$ . Consider  $\mathfrak{u}$  as a vector space with basis chosen so that  $\log(\mathbf{U}(\mathbb{Q})) \subseteq \mathfrak{u}(\mathbb{Q})$ . Because  $\theta'$  is commensurable with  $\mathbf{U}(\mathbb{Z})$ , there is a number  $k$  so that every vector in  $\mathfrak{u}(k\mathbb{Z}) \subseteq \log(\theta')$ . Identify  $\text{Aut}(\mathfrak{u})$  with a  $\mathbb{Q}$ -defined algebraic subgroup of  $\text{GL}_n(\mathbb{C})$  for some  $n$ . By [Seg83, Ch6, Prop4], the map

$$\phi \mapsto \log \circ \phi \circ \exp$$

identifies  $\text{Aut}(\mathbf{U})(\mathbb{Q})$  with the group of Lie algebra automorphisms  $\text{Aut}(\mathfrak{u}(\mathbb{Q}))$ . Given  $p \in P$ , let  $M_p \in \text{GL}_n(\mathbb{Q})$  denote the matrix for the induced action of  $p$  on  $\mathfrak{u}$ . Since  $P$  preserves  $\log(\theta')$ , there is some finite index subgroup  $P' \leq P$  so that  $M_p \in \text{GL}_n(\mathbb{Z})$  for all  $p \in P'$  (cf. [Seg83, Ch6, Lem6]). It suffices to find some  $P'' \leq P'$  so that

$$\log(f(p \cdot f^{-1})) \in \mathfrak{u}(k\mathbb{Z}) \text{ for all } p \in P''.$$

Let  $X = \log(f)$ . Given any  $p \in P$ , let  $Y_p = \log(p \cdot f^{-1}) = -M_p X$ . By the Baker-Campbell-Hausdorff formula (see, for example, [Seg83]), we have

$$\log(f(p \cdot f^{-1})) = X + Y_p + q(X, Y_p),$$

where  $q(X, Y_p)$  is a finite sum of iterated Lie brackets of  $X$  and  $Y_p$  weighted by rational numbers. We will show that there is a finite index subgroup  $P'' \leq P'$  so that both  $X + Y_p \in \mathfrak{u}(k\mathbb{Z})$  and  $q(X, Y_p) \in \mathfrak{u}(k\mathbb{Z})$  for all  $p \in P''$ .

If  $N$  is any natural number, let  $\pi_N : \text{GL}_n(\mathbb{Z}) \rightarrow \text{GL}_n(\mathbb{Z}/N\mathbb{Z})$  be the quotient map and define a finite index subgroup

$$P'(N) = \{p \in P' \mid M_p \in \ker(\pi_N)\} \leq P'.$$

Now let  $\ell$  be the least common multiple of denominators of coefficients of  $X$ . On the one hand, if  $p \in P'(k\ell)$  then  $X + Y_p \in \mathfrak{u}(k\mathbb{Z})$ . On the other hand, if  $p \in P'(\ell N)$  for some  $N$ , then we may write  $Y_p = Z_p - X$  for some vector  $Z_p \in \mathfrak{u}(N\mathbb{Z})$ . Using bilinearity of the Lie bracket, we may write  $q(X, Y_p)$  as a finite sum of iterated Lie brackets of  $X$  and  $Z_p$  weighted by rational numbers. Because the structure constants of  $\mathfrak{u}$  are rational, it follows that there is some  $N_1$  large enough that  $q(X, Y_p) \in \mathfrak{u}(k\mathbb{Z})$  if  $p \in P'(k\ell N_1)$ . The lemma follows by setting  $P'' = P'(k\ell N_1)$ .  $\square$

*Proof of Theorem 6.1.* Given  $\Gamma$  and  $\mathbf{H}$  as in the theorem, let  $\mathbf{U} = \mathbf{U}_{\mathbf{H}}$ . Find a strongly polycyclic subgroup  $\Lambda \leq \mathbf{H}(\mathbb{Q})$  abstractly commensurable with  $\Gamma$  with  $\mathbf{T}$ ,  $\mathbf{D}$ ,  $C$ , and  $\theta$  as in Proposition 5.15. That is,  $\mathbf{T}$  is a maximal  $\mathbb{Q}$ -defined torus,  $\mathbf{D}$  is the centralizer of  $\mathbf{T}$  with  $C = \Lambda \cap \mathbf{D}$  a Zariski-dense subgroup, and  $\theta \leq \mathbf{U}(\mathbb{Q})$  is a good unipotent shadow of  $\Lambda$ .

By Corollary 5.12 there is an embedding

$$\xi : \text{Comm}(\Gamma) \rightarrow \text{Aut}_{\mathbb{Q}}(\mathbf{H}).$$

By definition of  $\text{Comm}_{\mathbf{H}|\mathbf{F}}(\Gamma)$ , this restricts to an embedding

$$\hat{\xi} : \text{Comm}_{\mathbf{H}|\mathbf{F}}(\Gamma) \rightarrow (\mathcal{A}_{\mathbf{H}|\mathbf{F}})_{\mathbb{Q}}.$$

There is a decomposition  $(\mathcal{A}_{\mathbf{H}|\mathbf{F}})_{\mathbb{Q}} = \text{Inn}_{\mathbf{F}}^{\mathbf{H}}(\mathbb{Q}) \cdot \mathcal{A}_{\mathbf{T}}^1(\mathbb{Q})$  by Lemma 5.18. We have only to show that both  $\text{Inn}_{\mathbf{F}}^{\mathbf{H}}(\mathbb{Q})$  and  $\mathcal{A}_{\mathbf{T}}^1(\mathbb{Q})$  are in the image of  $\hat{\xi}$ .

**Claim 1:**  $\text{Inn}_{\mathbf{F}}^{\mathbf{H}}(\mathbb{Q}) \leq \xi(\text{Comm}(\Gamma))$ .

*Proof of Claim 1:* Suppose  $\Phi \in \text{Inn}_{\mathbf{F}}^{\mathbf{H}}(\mathbb{Q})$ . Then there is some  $f \in \mathbf{F}(\mathbb{Q})$  so that  $\Phi(x) = fxf^{-1}$  for all  $x \in \mathbf{H}$ . Because  $\theta$  is Zariski-dense in  $\mathbf{U}$ , conjugation by  $f$  induces a commensuration of  $\theta$  by Theorem 4.2. Let  $\theta_1$  and  $\theta_2$  be finite index subgroups of  $\theta$  so that  $\Phi(\theta_1) = \theta_2$ . Let  $C' \leq C$  be a finite index subgroup normalizing both  $\theta_1$  and  $\theta_2$ . By Lemma 6.2, applied with  $\theta' = \theta_1 \cap \theta_2$  and  $P = C'$ , there is some finite index subgroup  $C'' \leq C'$  so that

$$(12) \quad fcf^{-1}c^{-1} \in \theta_1 \cap \theta_2$$

for all  $c \in C''$ . Because  $\mathbf{F}$  is normal in  $\mathbf{U}$ , for all  $c \in C''$  we have  $fcf^{-1}c^{-1} \in \mathbf{F}$ . By (12) and the fact that  $\theta \cap \mathbf{F} = \text{Fitt}(\Lambda)$ , for all  $c \in C''$  we have

$$(13) \quad fcf^{-1}c^{-1} \in \text{Fitt}(\Lambda) \cap \theta_1 \cap \theta_2.$$

Let  $F_1 = \theta_1 \cap \text{Fitt}(\Lambda)$  and  $F_2 = \theta_2 \cap \text{Fitt}(\Lambda)$ . Then  $\Phi$  induces an isomorphism  $F_1 \rightarrow F_2$ . Because  $C''$  normalizes both  $F_1$  and  $F_2$ , we may form subgroups  $\Lambda_1 = F_1C''$  and  $\Lambda_2 = F_2C''$ , both of which are of finite index in  $\Lambda$ . We claim that  $\Phi$  induces an isomorphism  $\Lambda_1 \rightarrow \Lambda_2$ . Suppose  $f_1 \in F_1$  and  $c_1 \in C''$ . Then  $ff_1f^{-1} \in F_2$  by definition of  $\theta_1$  and  $\theta_2$ , and  $fc_1f^{-1} = f_2c_1$  for some  $f_2 \in F_2$  by (13). Therefore

$$ff_1c_1f^{-1} = ff_1f^{-1}fc_1f^{-1} \in F_2C''.$$

It follows that  $\Phi$  induces an injection  $\Lambda_1 \rightarrow \Lambda_2$ . Note that (13) holds for all  $c \in C''$  with  $f$  replaced by  $f^{-1}$ . Similar reasoning then gives that  $\Phi^{-1}$  induces an injection  $\Lambda_2 \rightarrow \Lambda_1$ . Thus  $\Phi$  induces a partial automorphism  $\Lambda_1 \rightarrow \Lambda_2$  of  $\Lambda$ , and so induces a commensuration of  $\Gamma$ . This completes the proof of Claim 1.

**Claim 2:**  $\mathcal{A}_{\mathbf{T}}^1(\mathbb{Q}) \leq \xi(\text{Comm}(\Gamma))$ .

*Proof of Claim 2:* Suppose  $\Phi \in \mathcal{A}_{\mathbf{T}}^1(\mathbb{Q})$ . Then  $\Phi$  corresponds to a  $\mathbb{Q}$ -defined map under the restriction  $\mathcal{A}_{\mathbf{T}}^1 \rightarrow \text{Aut}(\mathbf{U})$ , so  $\Phi$  induces a partial automorphism  $\theta_1 \rightarrow \theta_2$  of  $\theta$  by Theorem 4.2. The map  $C \rightarrow C_u$  is a homomorphism.

Define a finite index subgroup

$$C_1 = \{c \in C \mid c_u \in \theta_1\} \leq C.$$

Take any  $c_1 \in C_1$ , and write  $c_1 = u_1 s$  for  $u_1 \in \theta_1$  and  $s \in \mathbf{T}$ . Since  $\Phi \in \mathcal{A}_{\mathbf{H}|\mathbf{F}}$ , there is some  $f \in \mathbf{F}(\mathbb{Q})$  so that  $\Phi(u_1) = f u_1$ . Since  $\Phi \in \mathcal{A}_{\mathbf{T}}^1$ , we have

$$\Phi(c_1) = \Phi(u_1)\Phi(s) = f u_1 s = f c_1.$$

Both  $\Phi(u_1)$  and  $u_1$  are in  $\theta$ , so  $f \in \theta \cap \mathbf{F} = \text{Fitt}(\Lambda)$ . Therefore  $\Phi(c_1) \in \Lambda$ . Since  $\Phi$  preserves  $\mathbf{T}$ , it also preserves  $\mathbf{D}$ . Therefore  $\Phi(C_1) \leq C$  since  $\Lambda \cap \mathbf{D} = C$ .

Define

$$C_2 = \{c \in C \mid c_u \in \theta_2\} \leq C.$$

It is evident from the definitions of  $\theta_1$  and  $\theta_2$  that  $\Phi(C_1) \leq C_2$ . Applying the same logic as above to  $\Phi^{-1}$ , we conclude that  $\Phi(C_1) = C_2$ . Therefore  $\Phi$  induces a partial automorphism  $C_1 \rightarrow C_2$  of  $C$ .

Since  $\Phi$  preserves  $\mathbf{F}$ , it induces a partial automorphism  $F_1 \rightarrow F_2$  of  $\text{Fitt}(\Lambda)$ . Without loss of generality, suppose  $F_1$  is characteristic in  $\text{Fitt}(\Lambda)$ . Then  $F_1 C_1$  and  $F_2 C_2$  are both finite index subgroups of  $\Lambda$ . So  $\Phi$  induces a partial automorphism  $F_1 C_2 \rightarrow F_2 C_2$  of  $\Lambda$ , and hence a commensuration of  $\Gamma$ . This completes the proof of Claim 2.

Claims 1 and 2 show that  $\hat{\xi}$  is surjective, and therefore  $\hat{\xi}$  exhibits an isomorphism  $\text{Comm}_{\mathbf{H}|\mathbf{F}}(\Gamma) \approx (\mathcal{A}_{\mathbf{H}|\mathbf{F}})_{\mathbb{Q}}$ . This completes the proof of Theorem 6.1.  $\square$

*Proof of Theorem 1.1:* Let  $\mathbf{H}$  be the virtual algebraic hull of  $\Gamma$ . By Corollary 5.12 there is an embedding

$$\xi : \text{Comm}(\Gamma) \rightarrow \text{Aut}(\mathbf{H})(\mathbb{Q}),$$

where  $\text{Aut}(\mathbf{H})$  has the structure of an algebraic group as described in Section 5.4. Let  $\mathcal{A}_{\Gamma}$  be the Zariski-closure of  $\xi(\text{Comm}(\Gamma))$  in  $\text{Aut}(\mathbf{H})$ . Then  $\mathcal{A}_{\Gamma}$  is a  $\mathbb{Q}$ -defined algebraic group, by Proposition 2.1. Now take any  $\Psi \in \mathcal{A}_{\Gamma}(\mathbb{Q})$ . Take any element  $\Phi \in \xi(\text{Comm}(\Gamma))$  so that  $\Psi \circ \Phi^{-1} \in \mathcal{A}_{\Gamma}^0(\mathbb{Q})$ . We have  $\mathcal{A}_{\Gamma}^0 \leq \mathcal{A}_{\mathbf{H}|\mathbf{U}}$  by Lemma 5.16 and then  $\mathcal{A}_{\Gamma}^0 \leq \mathcal{A}_{\mathbf{H}|\mathbf{F}}$  by Lemma 5.17. Therefore so  $\Psi \circ \Phi^{-1} \in \mathcal{A}_{\mathbf{H}|\mathbf{F}}(\mathbb{Q})$ . It follows from Theorem 6.1 that  $\Psi \in \xi(\text{Comm}(\Gamma))$ , hence the isomorphism

$$\text{Comm}(\Gamma) \approx \mathcal{A}_{\Gamma}(\mathbb{Q}).$$

We have only to show that the image of  $\text{Aut}(\Gamma)$  in  $\text{Aut}(\mathbf{H})$  is commensurable with  $\mathcal{A}_{\Gamma}(\mathbb{Z})$ . Let  $F = \text{Fitt}(\Gamma)$  and define

$$A_{\Gamma|F} = \left\{ \phi \in \text{Aut}(\Gamma) \mid \phi|_{\Gamma/F} = \text{Id}|_{\Gamma/F} \right\}.$$

The proof of Lemma 5.17 shows that  $A_{\Gamma|F}$  is finite index in  $\text{Aut}(\Gamma)$ ; see also [BK11, 8.9]. The group  $A_{\Gamma|F}$  is commensurable with  $\mathcal{A}_{\mathbf{H}|\mathbf{F}}(\mathbb{Z})$  by [BG06, 8.9], so the result follows.  $\square$

## 7. COMMENSURATIONS OF LATTICES IN SEMISIMPLE GROUPS

Abstract commensurators of lattices in semisimple Lie groups not isogenous to  $\mathrm{PSL}_2(\mathbb{R})$  are fairly well understood, by work of Borel, Mostow, and Margulis. For example, see the first section of [AB94]. We recall the basic results here for completeness.

### 7.1. Arithmetic lattices in semisimple groups.

**Definition 7.1.** Suppose  $\Gamma \leq S$  is a lattice in a semisimple Lie group with trivial center and no compact factors. We say that  $\Gamma$  is *arithmetic* if there is a  $\mathbb{Q}$ -defined semisimple algebraic group  $\mathbf{S}$  and a surjective homomorphism  $f : \mathbf{S}(\mathbb{R})^0 \rightarrow S$  with compact kernel such that  $f(\mathbf{S}(\mathbb{Z}) \cap \mathbf{S}(\mathbb{R})^0)$  and  $\Gamma$  are commensurable.

Note that  $\mathbf{S}$  may be chosen to be simply-connected, and that  $\Gamma \doteq \mathbf{S}(\mathbb{Z})$  by Proposition 3.8. Hence, to compute the abstract commensurators of arithmetic lattices in semisimple Lie groups, it suffices to consider groups of the form  $\mathbf{S}(\mathbb{Z})$  for a simply-connected  $\mathbb{Q}$ -defined semisimple algebraic group  $\mathbf{S}$ .

Now suppose  $\mathbf{S}$  is a  $\mathbb{Q}$ -defined, simply-connected, semisimple algebraic group, with Lie algebra  $\mathfrak{s}$ . There is an isomorphism (for example by [Mar91, 1.4.13]),

$$D : \mathrm{Aut}(\mathbf{S}) \rightarrow \mathrm{Aut}(\mathfrak{s}).$$

The Lie algebra  $\mathfrak{s}$  has a natural  $\mathbb{Q}$ -structure, and  $D(\Phi) \in \mathrm{Aut}(\mathfrak{s})$  is defined over  $\mathbb{Q}$  if and only if  $\Phi \in \mathrm{Aut}(\mathbf{S})$  is defined over  $\mathbb{Q}$ . The group  $\mathrm{Aut}(\mathfrak{s}) \leq \mathrm{GL}(\mathfrak{s})$  is a  $\mathbb{Q}$ -defined algebraic group such that

$$\mathrm{Aut}_{\mathbb{Q}}(\mathfrak{s}) = \mathrm{Aut}(\mathfrak{s})(\mathbb{Q}).$$

Under the identification  $D$ , the group  $\mathrm{Aut}(\mathbf{S})$  has the structure of a  $\mathbb{Q}$ -defined algebraic group.

Recall that a  $\mathbb{Q}$ -defined, connected, semisimple algebraic group  $\mathbf{S}$  is *without  $\mathbb{Q}$ -compact factors* if there is no nontrivial,  $\mathbb{Q}$ -defined, connected, normal subgroup  $\mathbf{N} \leq \mathbf{S}$  such that  $\mathbf{N}(\mathbb{R})$  is compact. Note that given any  $\mathbb{Q}$ -defined connected, simply-connected, semisimple algebraic group, there is a  $\mathbb{Q}$ -defined, connected, simply-connected, semisimple algebraic group  $\mathbf{S}'$  without  $\mathbb{Q}$ -compact factors such that  $\mathbf{S}(\mathbb{Z})$  and  $\mathbf{S}'(\mathbb{Z})$  are abstractly commensurable.

**Proposition 7.2.** *Suppose  $\mathbf{S}$  is a  $\mathbb{Q}$ -defined, connected, simply-connected, semisimple algebraic group without  $\mathbb{Q}$ -compact factors. Then there is a canonical inclusion*

$$\Xi : \mathrm{Aut}(\mathbf{S})(\mathbb{Q}) \hookrightarrow \mathrm{Comm}(\mathbf{S}(\mathbb{Z})).$$

*Proof.* If  $\Phi \in \mathrm{Aut}(\mathbf{S})(\mathbb{Q})$ , then  $\Phi$  is a  $\mathbb{Q}$ -defined automorphism of  $\mathbf{S}$ . It follows from [Rag72, 10.14] that  $\Phi$  induces a commensuration of  $\mathbf{S}(\mathbb{Z})$ . Because  $\mathbf{S}(\mathbb{Z})$  is Zariski-dense in  $\mathbf{S}$  by Theorem 2.10, the induced map  $\Xi : \mathrm{Aut}(\mathbf{S})(\mathbb{Q}) \rightarrow \mathrm{Comm}(\mathbf{S}(\mathbb{Z}))$  is injective.  $\square$



The following consequence of Mostow–Prasad–Margulis rigidity is likely known to experts. We include a proof, having found no reference in the literature, using the techniques of [GP99a].

**Theorem 7.3.** *Let  $\mathbf{S}$  be a  $\mathbb{Q}$ -defined, connected, simply-connected, semisimple algebraic group without  $\mathbb{Q}$ -compact factors. Suppose that if  $F$  is a factor of  $\mathbf{S}(\mathbb{R})^0$  locally isomorphic to  $\mathrm{PSL}_2(\mathbb{R})$  then  $\mathbf{S}(\mathbb{Z})$  projects to a non-discrete subgroup of  $F$ . Then the inclusion*

$$\Xi : \mathrm{Aut}(\mathbf{S})(\mathbb{Q}) \rightarrow \mathrm{Comm}(\mathbf{S}(\mathbb{Z}))$$

*is an isomorphism.*

*Proof.* Let  $\mathbf{S}_1, \dots, \mathbf{S}_n$  be the  $\mathbb{Q}$ -simple factors of  $\mathbf{S}$ , so that

$$\mathbf{S} = \mathbf{S}_1 \cdot \mathbf{S}_2 \cdot \dots \cdot \mathbf{S}_{n-1} \cdot \mathbf{S}_n.$$

For each  $j$ , let  $\pi_j : \mathbf{S} \rightarrow \mathbf{S}_j$  be the canonical projection.

Suppose  $[\phi] \in \mathrm{Comm}(\mathbf{S}(\mathbb{Z}))$ . Without loss of generality, we may assume that  $\phi : \Gamma_1 \rightarrow \Gamma_2$  is a partial isomorphism of  $\mathbf{S}(\mathbb{Z})$  where

$$\Gamma_1 = (\Gamma_1 \cap \mathbf{S}_1) \cdot (\Gamma_1 \cap \mathbf{S}_2) \cdot \dots \cdot (\Gamma_1 \cap \mathbf{S}_n).$$

Let

$$\Gamma_{1,i} = \Gamma_1 \cap \mathbf{S}_i \quad \text{and} \quad \Gamma_1^i = \Gamma_{1,1} \cdot \dots \cdot \Gamma_{1,i-1} \cdot \Gamma_{1,i+1} \cdot \dots \cdot \Gamma_{1,n}.$$

Note that each  $\Gamma_{1,i}$  is of finite index in  $\mathbf{S}_i(\mathbb{Z})$ .

Given any  $i$ , choose some  $j$  such that  $\pi_j(\phi(\Gamma_{1,i}))$  is nontrivial. Let  $\mathbf{A}_1$  be the Zariski closure of  $\pi_j(\phi(\Gamma_{1,i}))$  in  $\mathbf{S}_j$ , and  $\mathbf{A}_2$  be the Zariski closure of  $\pi_j(\phi(\Gamma_1^i))$  in  $\mathbf{S}_j$ . Replacing  $\Gamma_1$  with a finite index subgroup if necessary, we may assume both  $\mathbf{A}_1$  and  $\mathbf{A}_2$  are connected. Then  $\mathbf{A}_1$  commutes with  $\mathbf{A}_2$  because  $\Gamma_{1,i}$  commutes with  $\Gamma_1^i$ . Note that  $\pi_j(\phi(\Gamma_{1,i})) \cdot \pi_j(\phi(\Gamma_1^i))$  is commensurable with  $\mathbf{S}_j(\mathbb{Z})$ , hence Zariski-dense in  $\mathbf{S}_j$  by Theorem 2.10. Therefore  $\mathbf{A}_1 \cdot \mathbf{A}_2 = \mathbf{S}_j$ . Since  $\pi_j(\phi(\Gamma_{1,i}))$  is nontrivial and  $\mathbb{Q}$ -defined, and  $\mathbf{S}_j$  is  $\mathbb{Q}$ -simple, it must be that  $\mathbf{A}_1 = \mathbf{S}_j$ . Since  $\mathbf{A}_1$  commutes with  $\mathbf{A}_2$  and  $\mathbf{A}_2$  is connected, it follows that  $\mathbf{A}_2$  is trivial. Therefore  $\pi_j(\phi(\Gamma_1^i))$  must be trivial.

It follows that for each  $i$  there is exactly one  $j$  so that  $\pi_j(\phi(\Gamma_{1,i}))$  is nontrivial. Therefore for each  $i$  there is exactly one  $j$  so that the image of  $\Gamma_{1,i}$  under  $\phi$  is a subgroup of  $\mathbf{S}_j$  of finite index in  $\mathbf{S}_j(\mathbb{Z})$ . It follows from Theorem 2.12 that  $\phi|_{\Gamma_{1,i}}$  virtually extends to an isomorphism  $\Phi_i : \mathbf{S}_i \rightarrow \mathbf{S}_j$ .

The map  $\Phi : \mathbf{S} \rightarrow \mathbf{S}$  defined by  $\Phi|_{\mathbf{S}_i} = \Phi_i$  is a  $\mathbb{Q}$ -defined automorphism virtually extending  $\phi$ , and so  $\Xi$  is surjective.  $\square$

**7.2. More general lattices in semisimple groups.** A lattice  $\Gamma$  in a connected semisimple Lie group  $S$  with finite center is *irreducible* if the projection of  $\Gamma$  to  $S/N$  is dense for every nontrivial connected normal subgroup  $N \leq S$ . Let  $\Gamma \leq S$  be an irreducible lattice in a connected semisimple Lie group with trivial center and no compact factors. The relative commensurator  $\mathrm{Comm}_S(\Gamma)$  satisfies a dichotomy (see [Zim84]): either  $\mathrm{Comm}_S(\Gamma)$

contains  $\Gamma$  as a subgroup of finite index, or  $\text{Comm}_S(\Gamma)$  is dense in  $S$ . In fact, it is a celebrated theorem of Margulis that this is precisely the dichotomy of arithmeticity versus non-arithmeticity.

**Theorem 7.4** (Margulis, see [Zim84], [Mar91]). *Let  $\Gamma \leq S$  be an irreducible lattice in a connected semisimple Lie group with trivial center and no compact factors. Then  $\text{Comm}_S(\Gamma)$  is dense in  $S$  if and only if  $\Gamma$  is arithmetic.*

We summarize the above results:

**Theorem 7.5.** *Let  $\Gamma$  be an irreducible lattice in a noncompact connected semisimple Lie group  $S$ . Assume that  $S$  is not locally isomorphic to  $\text{PSL}_2(\mathbb{R})$ . One of the following holds:*

- (1)  $\Gamma$  is arithmetic and there is a  $\mathbb{Q}$ -defined, connected, simply-connected,  $\mathbb{Q}$ -simple, semisimple algebraic group  $\mathbf{S}$  so that

$$\text{Comm}(\Gamma) \approx \text{Aut}(\mathbf{S})(\mathbb{Q}).$$

*Moreover, the group  $\text{Aut}(\Gamma)$  is commensurable with  $\text{Aut}(\mathbf{S})(\mathbb{Z})$ .*

- (2)  $\Gamma$  is not arithmetic and  $\text{Comm}(\Gamma) \doteq \Gamma$ .

*Proof.* Suppose  $\Gamma$  is arithmetic. Then there is a  $\mathbb{Q}$ -defined, connected, simply-connected, semisimple algebraic group  $\mathbf{S}$  without  $\mathbb{Q}$ -compact factors so that  $\Gamma \doteq \mathbf{S}(\mathbb{Z})$ . Since  $\Gamma$  is irreducible in  $S$ , the group  $\mathbf{S}$  is  $\mathbb{Q}$ -simple. The isomorphism  $\text{Comm}(\Gamma) \approx \text{Aut}(\mathbf{S})(\mathbb{Q})$  follows from Theorem 7.3. Since  $\text{Aut}(\Gamma)$  is commensurable with  $\Gamma$  and  $\Gamma$  is commensurable with  $\mathbf{S}(\mathbb{Z})$ , the result follows since  $\mathbf{S}(\mathbb{Z})$  is commensurable with  $\text{Aut}(\mathbf{S})(\mathbb{Z})$ .

Now suppose  $\Gamma$  is not arithmetic. Let  $S' = S/Z(S)$  and  $\pi : S \rightarrow S'$  the canonical projection. There is a finite index subgroup of  $\Gamma$  taken faithfully to a lattice  $\Gamma' \leq S'$ . Let  $N$  be the maximal compact factor of  $S'$  and  $S'' = S'/N$ . Then  $\Gamma'$  contains a finite index subgroup  $\Gamma''$  mapping isomorphically to a lattice  $\Gamma'' \leq S''$ . By Mostow–Prasad–Margulis rigidity (c.f. [Mos73]), every commensuration of  $\Gamma''$  extends to an automorphism of  $S''$ . Since  $[\text{Aut}(S'') : \text{Inn}(S'')] < \infty$ , where  $\text{Inn}(S'')$  is the group of inner automorphisms of  $S''$ , it follows that  $[\text{Comm}(\Gamma'') : \text{Comm}_{S''}(\Gamma'')] < \infty$ , and hence  $[\text{Comm}(\Gamma'') : \Gamma''] < \infty$  by Theorem 7.4. Since  $\Gamma''$  is of finite index in  $\Gamma$ , the result follows.  $\square$

The case that  $S = \text{PSL}_2(\mathbb{R})$  is dramatically different.

**Proposition 7.6.** *Suppose  $S$  is locally isomorphic to  $\text{PSL}_2(\mathbb{R})$  and  $\Gamma \leq S$  is a lattice. Then there is no faithful embedding  $\text{Comm}(\Gamma) \rightarrow \text{GL}_N(\mathbb{C})$  for any  $N$ .*

*Proof.*  $\Gamma$  is either virtually free or virtually the fundamental group of a closed surface. All finitely generated free groups are abstractly commensurable to each other, as are all closed surface groups. Therefore we have that  $\text{Comm}(\Gamma)$  is isomorphic either to  $\text{Comm}(F_2)$  or to  $\text{Comm}(\pi_1(\Sigma_2))$ , where  $F_n$  is the free group on  $n$  letters and  $\Sigma_g$  is a closed surface of genus  $g$ .

A group  $G$  has the *unique root property* if  $x^k = y^k$  implies  $x = y$  for all  $x, y \in G$  and nonzero  $k$ . If  $G$  has the unique root property and  $H \leq G$

is a finite index subgroup, then the natural map  $\text{Aut}(H) \rightarrow \text{Comm}(G)$  is faithful (see [Odd05]). It is easy to see that free groups and closed surface groups have the unique root property. Therefore  $\text{Aut}(F_n) \leq \text{Comm}(F_2)$  for all  $n \geq 2$ , and  $\text{Aut}(\pi_1(\Sigma_g)) \leq \text{Comm}(\pi_1(\Sigma_2))$  for all  $g \geq 2$ .

In [FP92] it is shown that  $\text{Aut}(F_n)$  is not linear for any  $n \geq 3$ . Therefore  $\text{Comm}(F_2)$  cannot be linear. On the other hand, the proof of [FLM01, 1.6] shows that for each  $N$  there is some  $g_0$  so that if  $g \geq g_0$  then  $\text{Mod}^\pm(\Sigma_{g,1})$ , the extended mapping class group of the punctured surface of genus  $g$ , has no faithful complex linear representation of dimension less than or equal to  $N$ . Since  $\text{Mod}^\pm(\Sigma_{g,1}) \approx \text{Aut}(\pi_1(\Sigma_g))$ , it follows that  $\text{Comm}(\pi_1(\Sigma_2))$  is not linear.  $\square$

Nonarithmetic irreducible lattices can occur only in groups isogenous to  $\text{SO}(1, n)$  or  $\text{SU}(1, n)$  up to compact factors. We will use this fact in §8.

**Theorem 7.7** (see [Mar91], [GS92]). *Let  $S$  be a connected semisimple Lie group with trivial center and no compact factors. Suppose either  $S = \text{Sp}(1, n)$  for  $n \geq 2$ ,  $S = F_4^{-20}$ , or  $\mathbb{R}\text{-rank}(S) \geq 2$ . Then every irreducible lattice in  $S$  is arithmetic.*

**7.3. Example:**  $\text{PSL}_n(\mathbb{Z})$ . Let  $S = \text{PSL}_n(\mathbb{R})$  and  $\Gamma = \text{PSL}_n(\mathbb{Z})$  for  $n \geq 3$ . There is an isomorphism  $\text{Comm}(\Gamma) \approx \text{Comm}_S(\Gamma) \rtimes \langle \tau \rangle$ , where  $\tau$  is the order 2 automorphism given by  $\tau(A) = (A^{-1})^t$ . To compute  $\text{Comm}_S(\Gamma)$ , we apply the following theorem of Borel.

**Theorem 7.8** ([Bor66]). *Let  $\mathbf{S}$  be a  $\mathbb{Q}$ -defined, connected, semisimple algebraic group without  $\mathbb{Q}$ -compact factors and with trivial center. Then*

$$\text{Comm}_{\mathbf{S}}(\mathbf{S}(\mathbb{Z})) = \mathbf{S}(\mathbb{Q}).$$

To apply Borel's theorem it is necessary to understand the structure of  $\text{PSL}_n(\mathbb{C})$  as an algebraic group. The group  $\text{PSL}_n(\mathbb{C})$  acts faithfully by conjugation on  $M_{n \times n}(\mathbb{C})$ , the space of  $n \times n$  complex matrices. This action is by  $\mathbb{C}$ -algebra automorphisms. In fact this gives an isomorphism  $\text{PSL}_n(\mathbb{C}) \approx \text{Aut}(M_{n \times n}(\mathbb{C}))$  by the Skolem–Noether theorem. The group of algebra automorphisms of  $M_{n \times n}(\mathbb{C})$  is a  $\mathbb{Q}$ -defined subgroup of  $\text{GL}_{n^2}(\mathbb{C})$ . Let  $\mathbf{S}_n$  denote the group  $\text{PSL}_n(\mathbb{C})$  equipped with this structure as a  $\mathbb{Q}$ -defined algebraic group.

The quotient map  $\pi : \text{SL}_n(\mathbb{C}) \rightarrow \mathbf{S}_n$  is a  $\mathbb{Q}$ -defined map of  $\mathbb{Q}$ -defined algebraic groups. Note that  $\pi(\text{SL}_n(\mathbb{Z}))$  is not generally equal to  $\mathbf{S}_n(\mathbb{Z})$ . For example, the matrix  $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \text{SL}_2(\mathbb{C})$  projects to an element of  $\mathbf{S}_2(\mathbb{Z})$ . However, it suffices to show that  $\mathbf{S}_n(\mathbb{Z})$  is commensurable with  $\text{PSL}_n(\mathbb{Z})$ .

**Theorem 7.9** ([Bor66]). *Let  $\pi : \mathbf{A}_1 \rightarrow \mathbf{A}_2$  be a surjective  $\mathbb{Q}$ -defined morphism of algebraic groups. Then  $\pi(\mathbf{A}_1(\mathbb{Z}))$  is commensurable with  $\mathbf{A}_2(\mathbb{Z})$ .*

**Corollary 7.10.**  *$\text{PSL}_n(\mathbb{Z})$  is abstractly commensurable with  $\mathbf{S}_n(\mathbb{Z})$ .*

**Corollary 7.11.** *If  $n \geq 3$ , then*

$$\text{Comm}(\text{PSL}_n(\mathbb{Z})) \approx \mathbf{S}_n(\mathbb{Q}) \rtimes \langle \tau \rangle.$$

Beware! The group  $\mathbf{S}_n(\mathbb{Q})$  is *not* isomorphic to  $\mathrm{PSL}_n(\mathbb{Q})$  under the definition  $\mathrm{PSL}_n(\mathbb{Q}) = \mathrm{SL}_n(\mathbb{Q})/Z(\mathrm{SL}_n(\mathbb{Q}))$ . In fact,  $\mathbf{S}_n(\mathbb{Q})$  is much larger than  $\mathrm{PSL}_n(\mathbb{Q})$  for  $n \geq 2$ . This may be seen precisely by an argument using Galois cohomology. The kernel of  $\pi : \mathrm{SL}_n \rightarrow \mathbf{S}_n$  is isomorphic to the multiplicative group of order  $n$ , denoted  $\mu_n$ . As in [PR94, 2.2.3], the exact sequence of  $\mathbb{Q}$ -groups

$$1 \rightarrow \mu_n \rightarrow \mathrm{SL}_n \rightarrow \mathbf{S}_n \rightarrow 1$$

gives rise to a long exact sequence of cohomology groups

$$1 \rightarrow \mu_n(\mathbb{Q}) \rightarrow \mathrm{SL}_n(\mathbb{Q}) \rightarrow \mathbf{S}_n(\mathbb{Q}) \rightarrow H^1(\overline{\mathbb{Q}}/\mathbb{Q}, \mu_n).$$

There is an isomorphism  $H^1(\overline{\mathbb{Q}}/\mathbb{Q}, \mu_n) \approx \mathbb{Q}^*/(\mathbb{Q}^*)^n$ . This is infinitely generated for  $n \geq 2$ , hence  $[\mathbf{S}_n(\mathbb{Q}) : \mathrm{PSL}_n(\mathbb{Q})] = \infty$ . In general, there is an isomorphism

$$\mathbf{S}_n(\mathbb{Q}) \approx \left\{ \left[ \frac{A}{\sqrt[n]{\det(A)}} \right] \mid A \in \mathrm{GL}_n(\mathbb{Q}) \right\}.$$

## 8. COMMENSURATIONS OF GENERAL LATTICES

Suppose  $\Gamma$  is a lattice in a connected Lie group  $G$  which is not necessarily either solvable or semisimple. Our main result is:

**Theorem 1.2.** *Suppose  $G$  is a connected, linear Lie group with simply-connected solvable radical. Suppose  $\Gamma \leq G$  is a lattice with the property that there is no surjection  $\phi : G \rightarrow H$  to any group  $H$  locally isomorphic to any  $\mathrm{SO}(1, n)$  or  $\mathrm{SU}(1, n)$  so that  $\phi(\Gamma)$  is a lattice in  $H$ . Then:*

- (1)  $\Gamma$  *virtually embeds in a  $\mathbb{Q}$ -defined algebraic group  $\mathbf{G}$  with Zariski-dense image so that every commensuration  $[\phi] \in \mathrm{Comm}(\Gamma)$  induces a unique  $\mathbb{Q}$ -defined automorphism of  $\mathbf{G}$  virtually extending  $\phi$ .*
- (2) *There is a  $\mathbb{Q}$ -defined algebraic group  $\mathcal{B}$  so that*

$$\mathrm{Comm}(\Gamma) \approx \mathcal{B}(\mathbb{Q})$$

*and the image of  $\mathrm{Aut}(\Gamma)$  in  $\mathcal{B}$  is commensurable with  $\mathcal{B}(\mathbb{Z})$ .*

The proof of Theorem 1.2 proceeds in four steps:

- (1) Construct the algebraic group  $\mathbf{G}$ , called the *virtual algebraic hull* of  $\Gamma$ , such that  $\Gamma$  virtually embeds in  $\mathbf{G}$  with Zariski-dense image.
- (2) Show that commensurations of  $\Gamma$  induce  $\mathbb{Q}$ -defined automorphisms of  $\mathbf{G}$ .
- (3) Show that  $\mathrm{Aut}(\mathbf{G})$  has the structure of an algebraic group, and that  $\mathrm{Comm}(\Gamma)$  is realized as the  $\mathbb{Q}$ -points of a  $\mathbb{Q}$ -defined subgroup of  $\mathrm{Aut}(\mathbf{G})$ .
- (4) Show that the image of  $\mathrm{Aut}(\Gamma)$  in  $\mathrm{Aut}(\mathbf{G})$  is commensurable with  $\mathcal{B}(\mathbb{Z})$ .

*Proof of Theorem 1.2:* Let  $\Gamma$  be as in the theorem. Let  $R$  be the solvable radical of  $G$ .

**Step 1: (Construction of virtual algebraic hull).** We will construct  $\mathbf{G}$  as the semidirect product of a solvable group  $\mathbf{H}$  with a semisimple group  $\mathbf{S}$ . Roughly speaking,  $\mathbf{H}$  is the virtual algebraic hull of the “solvable part” of  $\Gamma$ , while  $\mathbf{S}$  is a  $\mathbb{Q}$ -defined semisimple group without  $\mathbb{Q}$ -compact factors such that the “semisimple part” of  $\Gamma$  is abstractly commensurable with  $\mathbf{S}(\mathbb{Z})$ . To make this precise, we modify the Lie group  $G$  and lattice  $\Gamma$  as follows.

Because  $G$  is linear, there is a connected semisimple subgroup  $S \leq G$  so that  $G = R \rtimes S$ . Let  $\mathbf{S}$  be a  $\mathbb{Q}$ -defined linear algebraic group so that  $S = \mathbf{S}'(\mathbb{R})^0$ . There is a simply-connected algebraic group  $\tilde{\mathbf{S}}'$  and a surjection  $\pi : \tilde{\mathbf{S}}' \rightarrow \mathbf{S}'$  with finite central kernel. Let  $\tilde{S} = \tilde{\mathbf{S}}'(\mathbb{R})^0$ . Then  $\pi : \tilde{S} \rightarrow S$  is a finite covering map with central kernel. The lattice  $\Gamma_s \leq S$  lifts to a lattice  $\tilde{\Gamma}_s \leq \tilde{S}$ . Replacing  $S$  by  $\tilde{S}$  and  $\Gamma$  by  $\tilde{\Gamma}$ , we may assume that  $G$  is algebraically simply-connected.

First let  $K$  be the maximal compact quotient of  $S$  such that  $\Gamma$  projects to a finite subgroup of  $K$ . Because  $G$  is algebraically simply-connected,  $K$  may be identified with a subgroup of  $S$ , and there is a subgroup  $S' \leq S$  so that  $S = S' \times K$ . Then  $\Gamma \cap S'$  is of finite index in  $\Gamma$ , so we may replace  $S$  by  $S'$  and assume that  $\Gamma$  projects densely into the maximal compact factor of  $S$ . It follows by [Sta02, 4.5] that, passing to a finite index subgroup of  $\Gamma$ , we have chosen  $S \leq G$  so that  $\Gamma = (\Gamma \cap R)(\Gamma \cap S)$ . Let  $\Gamma_r = \Gamma \cap R$  and  $\Gamma_s = \Gamma \cap S$ . This makes precise our notions of “solvable” and “semisimple” parts of  $\Gamma$ .

We now want to find a  $\mathbb{Q}$ -defined algebraic group  $\mathbf{S}$  without  $\mathbb{Q}$ -compact factors so that  $\Gamma_s$  is abstractly commensurable with  $\mathbf{S}(\mathbb{Z})$ . Because  $S$  is algebraically simply-connected, there is a decomposition  $S = S_1 \times \cdots \times S_k$  so that  $\Gamma_s$  virtually decomposes as  $\Gamma_{s,1} \times \cdots \times \Gamma_{s,k}$ , where  $\Gamma_{s,i} \leq S_i$  is an irreducible lattice for each  $i$ . Since each  $\Gamma_{s,i}$  does not project to a lattice in  $\mathrm{SO}(1, n)$  or  $\mathrm{SU}(1, n)$ , it follows from Theorem 7.7 that for each  $i$  there is a connected  $\mathbb{Q}$ -defined semisimple algebraic group  $\mathbf{S}_i$  and a surjection  $\pi_i : \mathbf{S}_i(\mathbb{R})^0 \rightarrow S_i$  with compact kernel so that  $\pi_i(\mathbf{S}_i(\mathbb{Z}) \cap \mathbf{S}_i(\mathbb{R})^0)$  is commensurable with  $\Gamma_{s,i}$ . Set

$$\mathbf{S} = \mathbf{S}_1 \times \cdots \times \mathbf{S}_k \quad \text{and} \quad \Gamma'_s = \prod_{i=1}^k \mathbf{S}_i(\mathbb{Z}) \cap \mathbf{S}_i(\mathbb{R})^0.$$

Each  $\mathbf{S}_i$  is  $\mathbb{Q}$ -simple and  $\mathbf{S}_i(\mathbb{R})^0$  is not compact, so  $\mathbf{S}$  is without  $\mathbb{Q}$ -compact factors.

Our next goal is to define an action of  $\mathbf{S}$  the virtual algebraic hull of  $\Gamma_r$ . To do this, we use the fact that the virtual algebraic hull of  $\Gamma_r$  is a real algebraic hull for any unipotently connected, simply-connected solvable Lie group  $R$  containing  $\Gamma_r$  as a Zariski-dense lattice. A classical construction may be used to produce a simply-connected solvable Lie group  $R'$  so that  $\Gamma_r$  is Zariski-dense in  $R'$  and  $R'$  is unipotently connected. To ensure that

we can apply this construction while respecting the action of  $S$ , we present a proof based on ideas in [BK11].

**Lemma 8.1.** *Suppose  $G = R \rtimes S$  is an algebraically simply-connected Lie group with  $R$  solvable and  $S$  semisimple. Let  $\Gamma = (\Gamma \cap R)(\Gamma \cap S)$  be a lattice, and set  $\Gamma_r = \Gamma \cap R$  and  $\Gamma_s = \Gamma \cap S$ . There is a finite index subgroup  $\Gamma' \leq \Gamma$  of the form  $\Gamma' = \Gamma'_r \rtimes \Gamma_s$  and a simply-connected solvable Lie group  $R'$  so that  $\Gamma'$  is a lattice in  $R' \rtimes S$ , with the property that  $\Gamma'_r$  is Zariski-dense in  $R'$  and  $R'$  is unipotently connected.*

*Proof.* Let  $\mathbf{H}_R$  be the real algebraic hull of  $R$  and  $\mathbf{H}_\Gamma$  the virtual algebraic hull of  $\Gamma_r$ . There is a finite index characteristic subgroup  $\Gamma'_r \leq \Gamma_r$  so that  $\mathbf{H}_\Gamma$  is the algebraic hull of  $\Gamma'_r$ . By [BK11, 5.3] we may moreover assume that there is some simply-connected solvable Lie group  $R'$  that is unipotently connected and so that  $\Gamma'_r$  is Zariski-dense in  $R'$ . The algebraic group  $\mathbf{H}_\Gamma$  is a real algebraic hull for  $R'$  by [BK11, 3.11]. In particular, we identify  $R'$  with a subgroup  $R' \leq \mathbf{H}_\Gamma(\mathbb{R})$  containing  $\Gamma'_r$ .

By [BK11, 3.9], the inclusion  $\Gamma'_r \leq R$  extends to an  $\mathbb{R}$ -defined embedding  $\mathbf{H}_\Gamma \rightarrow \mathbf{H}_R$ . The action of  $S$  on  $R$  extends to an action of  $S$  on  $\mathbf{H}_R$  by  $\mathbb{R}$ -defined algebraic automorphisms. Let  $\Phi$  be an  $\mathbb{R}$ -defined automorphism of  $\mathbf{H}_R$  induced by some  $s \in S$ . We would like to show that  $\Phi$  preserves  $R'$ .

Let  $N$  be the maximal connected nilpotent normal subgroup of  $R$ , and let  $\mathbf{F}$  denote the Zariski-closure of  $\text{Fitt}(\Gamma)$  in  $\mathbf{H}_R$ . We clearly have  $N \leq \mathbf{F}$ . It follows from [BK11, 3.3] that  $N \leq \mathbf{H}_R(\mathbb{R})$  is normal. Because  $S$  is connected, the action of  $S$  on  $R/N$  is trivial by [BK11, 6.9]. It follows that  $\Phi(\mathbf{F}) = \mathbf{F}$ . By density of  $R \leq \mathbf{H}_R$ , we conclude that  $\Phi$  is trivial on the quotient  $\mathbf{H}_R/\mathbf{F}$ .

Let  $N'$  be the maximal normal nilpotent subgroup of  $R'$ . Then  $\mathbf{F}(\mathbb{R}) = N'$  in  $\mathbf{H}_\Gamma$  because  $R'$  is unipotently connected. It follows that  $\Phi(R') \leq R'\mathbf{F}(\mathbb{R}) = R'$ , and so  $\Phi$  induces an automorphism of  $R'$ . This agrees with the given action of  $\Gamma_s$  on  $\Gamma'_r$ , so we may form the semidirect product  $G' = R' \rtimes S$  containing the lattice  $\Gamma' = \Gamma'_r \rtimes \Gamma_s$ .  $\square$

We may therefore assume that the radical  $R$  of  $G$  is unipotently connected and  $\Gamma_r$  is Zariski-dense in  $R$ . Let  $\mathbf{H}$  be the virtual algebraic hull of  $\Gamma_r$ . Because  $R$  is unipotently connected and  $\Gamma_r$  is Zariski-dense in  $R$ , [BK11, 3.11] implies that  $\mathbf{H}$  has the structure of a  $\mathbb{R}$ -defined connected algebraic hull of  $R$ . There is a representation  $\rho : S \rightarrow \text{Aut}_{\mathbb{R}}(\mathbf{H})$  by the automorphism extension property of the algebraic hull. Because  $\mathbf{S}$  is simply-connected,  $\rho$  extends to an  $\mathbb{R}$ -defined representation  $\rho : \mathbf{S} \rightarrow \text{Aut}(\mathbf{H})$  by Proposition 2.9. Since  $\Gamma_s$  preserves  $\Gamma_r$ , we have that  $\rho(\Gamma)$  is  $\mathbb{Q}$ -defined for every  $\gamma \in \Gamma_s$ . Because  $\mathbf{S}$  is without  $\mathbb{Q}$ -compact factors and connected, we know  $\Gamma_s$  is Zariski-dense in  $\mathbf{S}$  by Theorem 2.10. It follows that the representation  $\rho : \mathbf{S} \rightarrow \text{Aut}(\mathbf{H})$  is  $\mathbb{Q}$ -defined.

The definition of the variety structure on  $\text{Aut}(\mathbf{H})$  implies that the action map  $\alpha : \mathbf{H} \times \text{Aut}(\mathbf{H}) \rightarrow \mathbf{H}$  is a  $\mathbb{Q}$ -defined map of varieties. It follows that

the action map  $\mathbf{H} \times \mathbf{S} \rightarrow \mathbf{H}$  is  $\mathbb{Q}$ -defined. The semidirect product of groups

$$(14) \quad \mathbf{G} = \mathbf{H} \rtimes \mathbf{S}$$

therefore has the structure of a  $\mathbb{Q}$ -defined algebraic group. It is evident from the construction that  $\Gamma$  embeds in  $\mathbf{G}(\mathbb{Q})$  as a Zariski-dense subgroup. This concludes the first step of the proof.

**Step 2: (Extension of commensurations).** We now construct a map

$$\xi : \text{Comm}(\Gamma) \rightarrow \text{Aut}_{\mathbb{Q}}(\mathbf{G}).$$

Let  $\Lambda$  be a thickening of  $\Gamma_r$  in  $\mathbf{H}$  with nilpotent supplement  $C$  and good unipotent shadow  $\theta$ , as in Proposition 5.15. The action of  $\Gamma_s$  on  $\Gamma_r$  extends to an action on  $\Lambda$ . Then  $\Lambda \rtimes \Gamma_s$  is a Zariski-dense subgroup of  $\mathbf{G}(\mathbb{Q})$  containing  $\Gamma$  as a finite index subgroup.

**Lemma 8.2.** *Let  $\mathbf{U}$  denote the unipotent radical of  $\mathbf{H}$ . Suppose  $u \in \mathbf{U}(\mathbb{Q})$ . Then conjugation by  $u$  induces a commensuration of  $\Gamma$ .*

*Proof.* Suppose  $u \in \mathbf{U}(\mathbb{Q})$ . Let  $\mathbf{F} = \text{Fitt}(\mathbf{H})$ . Conjugation by  $u$  induces two partial automorphisms: a partial automorphism  $\phi_\theta : \theta_1 \rightarrow \theta_2$  of  $\theta$ , and a partial automorphism  $\phi_R : \Lambda_1 \rightarrow \Lambda_2$  of  $\Gamma_r$  by Theorem 6.1. As in the proof of Theorem 6.1, we may choose  $\theta_1$ ,  $\theta_2$ ,  $\Lambda_1$ , and  $\Lambda_2$  so that  $\theta_i \cap \mathbf{F} = \text{Fitt}(\Lambda_i)$  for  $i = 1, 2$ . We want to find some finite index subgroup  $\Gamma_s'' \leq \Gamma_s$  so that conjugation by  $u$  induces an isomorphism  $\Lambda_1 \Gamma_s'' \rightarrow \Lambda_2 \Gamma_s''$ .

Let  $N$  be the maximal connected normal nilpotent subgroup of  $R$ . Because  $S$  is connected, Lie theory gives that the action of  $S$  on  $R$  is trivial on  $R/N$  (see [BK11, 6.9]). Since  $N$  is Zariski-dense in the Fitting subgroup  $\mathbf{F} \leq \mathbf{H}$  by [BK11, 5.4], the induced action of  $\Gamma_s$  on  $\mathbf{H}$  is trivial on the quotient  $\mathbf{H}/\mathbf{F}$ . Therefore for any  $s \in \Gamma_s$  we have

$$(15) \quad sus^{-1}u^{-1} \in \mathbf{F}.$$

Restricting our attention to  $\Lambda$ , we see that for any  $s \in \Gamma_s$  and  $c \in C$ , there is some  $f \in \text{Fitt}(\Lambda)$  so that  $scs^{-1} = fc$ . It follows that conjugation by  $s \in \Gamma_s$  preserves  $\theta$ . Let  $\Gamma_s' \leq \Gamma$  be a finite index subgroup normalizing both  $\Lambda_1$  and  $\Lambda_2$ . Then  $\Gamma_s'$  also normalizes both  $\theta_1$  and  $\theta_2$ . By Lemma 6.2, there is a finite index subgroup  $\Gamma_s'' \leq \Gamma_s'$  so that  $usu^{-1}s^{-1} \in \theta_1 \cap \theta_2$  for all  $s \in \Gamma_s''$ . Combining this with (15), for all  $s \in \Gamma_s''$  we have

$$(16) \quad usu^{-1}s^{-1} \in \text{Fitt}(\Lambda_1) \cap \text{Fitt}(\Lambda_2).$$

The same arguments as in Claim 1 of the proof of Theorem 6.1 show that conjugation by  $u$  induces a partial isomorphism  $\Lambda_1 \Gamma_s'' \rightarrow \Lambda_2 \Gamma_s''$  of  $\Lambda \rtimes \Gamma_s$ .  $\square$

**Proposition 8.3.** *Every commensuration  $[\phi] \in \text{Comm}(\Gamma)$  induces a unique  $\mathbb{Q}$ -defined automorphism of  $\mathbf{G}$  virtually extending  $\phi$ . Hence there is an injective homomorphism*

$$\xi : \text{Comm}(\Gamma) \rightarrow \text{Aut}_{\mathbb{Q}}(\mathbf{G}).$$

*Proof.* Suppose there are finite index subgroups  $\Gamma_1$  and  $\Gamma_2$  of  $\Lambda \rtimes \Gamma_s$  with  $\phi : \Gamma_1 \rightarrow \Gamma_2$  a partial automorphism representing  $[\phi]$ . Passing to a finite index subgroup so that  $\Gamma_s \cap Z(S)$  is trivial, we may assume that  $\Gamma_i \cap \mathbf{H}$  is the unique maximal normal solvable subgroup of  $\Gamma_i$  for  $i = 1, 2$  (cf. [Pra76, Lemma 6]). It follows that  $\phi(\Gamma_1 \cap \mathbf{H}(\mathbb{R})) = \Gamma_2 \cap \mathbf{H}(\mathbb{R})$ , and so  $\phi$  induces a commensuration  $[\phi_R] \in \text{Comm}(\Lambda)$  by Lemma 3.5. It follows from Corollary 5.12 that  $\phi_R$  extends to an automorphism  $\Phi_R \in \text{Aut}_{\mathbb{Q}}(\mathbf{H})$ .

Now let  $\mathbf{L}$  be the Zariski-closure of  $\phi(\Gamma_1 \cap \Gamma_s)$  in  $\mathbf{G}$ . Then  $\mathbf{L}$  is  $\mathbb{Q}$ -defined, and is semisimple by [Sta02, Theorem 2]. (Note that here we are using the assumption that  $\Gamma_s$  does not surject to a lattice in any  $\text{SU}(1, n)$  or  $\text{SO}(1, n)$ .) There is some  $u \in \mathbf{U}(\mathbb{Q})$  conjugating  $\mathbf{L}$  into  $\mathbf{S}$  by Theorem 2.6. It follows from Lemma 8.2 that  $\text{Inn}_u \circ \phi$  virtually restricts to a partial automorphism  $\phi_S : \Delta_1 \rightarrow \Delta_2$  of  $\Gamma_s$ . The partial automorphism  $\phi_S$  virtually extends to a  $\mathbb{Q}$ -defined automorphism  $\Phi_S \in \text{Aut}_{\mathbb{Q}}(\mathbf{S})$  by Theorem 7.3.

Define an automorphism  $\Phi \in \text{Aut}(\mathbf{G})$  by

$$\Phi(r, s) = \text{Inn}_{u^{-1}}(\Phi_R(r), \Phi_S(s)).$$

Then  $\Phi$  virtually extends the partial automorphism  $\phi$ . This extension is unique up to choice of  $u \in \mathbf{U}(\mathbb{Q})$  conjugating  $\mathbf{L}$  to  $\mathbf{S}$ . However, any two such  $u$  differ by an element of  $\mathbf{U}(\mathbb{Q})$  centralized by  $\mathbf{S}$ , hence  $\Phi$  is unique.  $\square$

**Step 3: (Algebraic structure).** We now show that the image of  $\xi : \text{Comm}(\Gamma) \rightarrow \text{Aut}_{\mathbb{Q}}(\mathbf{G})$  has the structure of the  $\mathbb{Q}$ -rational points of a  $\mathbb{Q}$ -defined algebraic group. We first show that  $\text{Aut}(\mathbf{G})$  in fact has the structure of a  $\mathbb{Q}$ -defined algebraic group.

**Definition 8.4.** A pair of automorphisms  $(\Phi_R, \Phi_S) \in \text{Aut}(\mathbf{H}) \times \text{Aut}(\mathbf{S})$  is *compatible* if there is some  $\Phi \in \text{Aut}(\mathbf{G})$  preserving  $\mathbf{S}$  with  $\Phi|_{\mathbf{H}} = \Phi_R$  and  $\Phi|_{\mathbf{S}} = \Phi_S$ . Let  $C(\mathbf{G}) \subseteq \text{Aut}(\mathbf{H}) \times \text{Aut}(\mathbf{S})$  be the set of compatible pairs of automorphisms.

As both  $\text{Aut}(\mathbf{H})$  and  $\text{Aut}(\mathbf{S})$  have structures of  $\mathbb{Q}$ -defined algebraic groups, their product  $\text{Aut}(\mathbf{H}) \times \text{Aut}(\mathbf{S})$  is a  $\mathbb{Q}$ -defined algebraic group.

**Lemma 8.5.**  $C(\mathbf{G})$  is a  $\mathbb{Q}$ -defined subgroup of  $\text{Aut}(\mathbf{H}) \times \text{Aut}(\mathbf{S})$ .

*Proof.* Let  $\rho : \mathbf{S} \rightarrow \text{Aut}(\mathbf{H})$  be the  $\mathbb{Q}$ -defined representation by conjugation. Any automorphism  $\Phi \in \text{Aut}(\mathbf{G})$  preserving  $\mathbf{S}$  must satisfy

$$[\Phi \circ \rho(s)](r) = \Phi(srs^{-1}) = \Phi(s)\Phi(r)\Phi(s)^{-1} = [\rho(\Phi(s)) \circ \Phi](r)$$

for all  $r \in \mathbf{H}$  and all  $s \in \mathbf{S}$ . From this it is clear that any  $(\Phi_R, \Phi_S) \in C(\mathbf{G})$  satisfies

$$(17) \quad \Phi_R \circ \rho(s) \circ \Phi_R^{-1} \circ \rho(\Phi_S(s))^{-1} = \text{Id} \in \text{Aut}(\mathbf{H})$$

for all  $s \in \mathbf{S}$ . Conversely, suppose a pair  $(\Phi_R, \Phi_S) \in \text{Aut}(\mathbf{H}) \times \text{Aut}(\mathbf{S})$  satisfies (17) for all  $s \in \mathbf{S}$ . Then the function  $\Phi : \mathbf{G} \rightarrow \mathbf{G}$  defined by  $\Phi(r, s) = \Phi_R(r)\Phi_S(s)$  is an automorphism of  $\mathbf{G}$ , and so  $(\Phi_R, \Phi_S) \in C(\mathbf{G})$ . Thus  $C(\mathbf{G})$  is equal to the set of pairs  $(\Phi_R, \Phi_S)$  satisfying (17) for all  $s \in \mathbf{S}$ .



For a fixed element  $s \in \mathbf{S}$ , the solution set of equation (17) is a  $\mathbb{Q}$ -defined closed subset of  $\text{Aut}(\mathbf{H}) \times \text{Aut}(\mathbf{S})$ . It follows that  $C(\mathbf{G})$  is a  $\mathbb{Q}$ -defined subgroup.  $\square$

**Lemma 8.6.** *The map*

$$(18) \quad \begin{aligned} \Theta : \mathbf{U} \rtimes C(\mathbf{G}) &\rightarrow \text{Aut}(\mathbf{G}) \\ (u, \Phi_R, \Phi_S) &\mapsto \text{Inn}_u \circ \Phi_R \circ \Phi_S \end{aligned}$$

*is a surjective group homomorphism with  $\mathbb{Q}$ -defined unipotent kernel. Hence  $\text{Aut}(\mathbf{G})$  has the structure of a  $\mathbb{Q}$ -defined algebraic group, such that*

$$(19) \quad \text{Aut}_{\mathbb{Q}}(\mathbf{G}) \approx \text{Aut}(\mathbf{G})(\mathbb{Q}) \approx \mathbf{U}(\mathbb{Q}) \rtimes C(\mathbf{G})(\mathbb{Q}) / (\ker \Theta)(\mathbb{Q}).$$

*Proof.* There is a natural action  $\alpha : C(\mathbf{G}) \times \mathbf{U} \rightarrow \mathbf{U}$  defined by

$$\alpha(\Phi_R, \Phi_S, u) = \Phi_R(u).$$

Then  $\alpha$  is a  $\mathbb{Q}$ -defined morphism of varieties, so the semidirect product  $\mathbf{U} \rtimes C(\mathbf{G})$  has the structure of a  $\mathbb{Q}$ -defined algebraic group. Surjectivity of  $\Theta$  follows from Theorem 2.6. One can easily check that

$$\ker(\Theta) = \{(u, \text{Inn}_{u^{-1}}, \text{Id}) \in \mathbf{U} \rtimes C(\mathbf{G}) \mid u \text{ is centralized by } \mathbf{S}\}.$$

Since the action of  $\mathbf{S}$  on  $\mathbf{H}$  is  $\mathbb{Q}$ -defined,  $\ker(\Theta)$  is a  $\mathbb{Q}$ -defined unipotent subgroup of  $\mathbf{U} \rtimes C(\mathbf{G})$ . Hence the quotient  $\mathbf{U} \rtimes C(\mathbf{G}) / \ker(\Theta)$  is a  $\mathbb{Q}$ -defined algebraic group. The former isomorphism of (19) follows from Theorem 2.6 and the definitions of the  $\mathbb{Q}$ -structures on  $\text{Aut}(\mathbf{H})$  and  $\text{Aut}(\mathbf{G})$ , and the latter follows from the standard arguments of [PR94, 2.2.3].  $\square$

We will now show that the image of

$$\xi : \text{Comm}(\Gamma) \rightarrow \text{Aut}(\mathbf{G})$$

is equal to the  $\mathbb{Q}$ -points of a  $\mathbb{Q}$ -defined subgroup of  $\text{Aut}(\mathbf{G})$ . Let  $\mathcal{A}_{\Gamma_r} \leq \text{Aut}(\mathbf{H})$  be the  $\mathbb{Q}$ -defined subgroup such that  $\mathcal{A}_{\Gamma_r}(\mathbb{Q}) \approx \text{Comm}(\Gamma_r)$ , as in Theorem 1.1. Define

$$\mathcal{B} = \{\Phi \in \text{Aut}(\mathbf{G}) \mid \Phi|_{\mathbf{H}} \in \mathcal{A}_{\Gamma_r}\}.$$

Then  $\mathcal{B}$  is evidently a  $\mathbb{Q}$ -defined subgroup of  $\text{Aut}(\mathbf{G})$ . It is clear that  $\xi(\text{Comm}(\Gamma)) \leq \mathcal{B}(\mathbb{Q})$ .

**Proposition 8.7.** *The map  $\xi : \text{Comm}(\Gamma) \rightarrow \mathcal{B}(\mathbb{Q})$  is an isomorphism.*

*Proof.* Clearly  $\xi$  is injective. Suppose  $\Phi \in \mathcal{B}(\mathbb{Q})$ . By Theorem 2.6 there is some  $u \in \mathbf{U}(\mathbb{Q})$  such that  $\text{Inn}_u \circ \Phi$  preserves  $\mathbf{S}$ . Since  $\text{Inn}_u \in \mathcal{A}_{\Gamma_r}$ , it follows that  $\text{Inn}_u \circ \Phi \in \mathcal{B}(\mathbb{Q})$ . Therefore there are  $\Phi_R \in \mathcal{A}_{\Gamma_r}(\mathbb{Q})$  and  $\Phi_S \in \text{Aut}(\mathbf{S})(\mathbb{Q})$  such that  $\text{Inn}_u \circ \Phi = \Phi_R \circ \Phi_S$ .

We have that  $\Phi_R$  induces a partial automorphism  $\phi_R : \Lambda_1 \rightarrow \Lambda_2$  of  $\Lambda$  by Theorem 1.1, and  $\Phi_S$  induces a partial automorphism  $\phi_S : \Gamma_{s,1} \rightarrow \Gamma_{s,2}$  of  $\Gamma_s$  by Proposition 7.2. We may choose  $\Lambda_1$  to be characteristic in  $\Lambda$ , and then choose  $\Gamma_{s,2}$  to normalize  $\Lambda_2 \leq \Lambda$ . It follows that there is a well-defined isomorphism  $\phi : \Lambda_1 \Gamma_{s,1} \rightarrow \Lambda_2 \Gamma_{s,2}$  defined by  $\phi(r, s) = \Phi_R(r) \Phi_S(s)$ , which

clearly satisfies  $\xi([\phi]) = \Phi_R \circ \Phi_S$ . Since  $\text{Inn}_u \in \xi(\text{Comm}(\Gamma))$  by Lemma 8.2, it follows that  $\Phi \in \xi(\text{Comm}(\Gamma))$ .  $\square$

**Step 4: ( $\text{Aut}(\Gamma)$  commensurable with  $\mathcal{B}(\mathbb{Z})$ ).** It remains only to show that  $\text{Aut}(\Gamma)$  is commensurable with  $\mathcal{B}(\mathbb{Z})$ . For this, we first show that the element  $u \in \mathbf{U}(\mathbb{Q})$  arising in the proof of Proposition 8.3 can be chosen in a controlled way. Given a vector space  $V$  of finite dimension, a subset  $L \subseteq V$  is a *vector space lattice* if  $L$  is a finitely generated  $\mathbb{Z}$ -submodule spanning  $V$ .

**Lemma 8.8.** *Let  $P$  be any group acting nontrivially and irreducibly on a vector space  $V \approx \mathbb{R}^n$ . Suppose  $P$  preserves a vector space lattice  $L' \subseteq V$ . Then there is a vector space lattice  $L \subseteq V$  such that if  $v - p \cdot v \in L'$  for all  $p \in P$ , then  $v \in L$ .*

*Proof.* The action of  $P$  descends to an action of  $P$  on the torus  $V/L'$ . It suffices to show that this action has finitely many fixed points, as these fixed points lift to the desired vector space lattice  $L \subseteq V$ . To see this, simply note that the fixed point set  $X$  of the action of  $P$  is a closed, hence compact, Lie subgroup of  $V/L'$ . The dimension of  $X$  must be zero by the assumption that  $P$  acts irreducibly and nontrivially on  $V$ . Therefore  $X$  is finite.  $\square$

**Lemma 8.9.** *There is a subgroup  $\Lambda \leq \mathbf{U}(\mathbb{Q})$  commensurable with  $\mathbf{U}(\mathbb{Z})$  such that if  $\phi \in \text{Aut}(\Gamma)$  extends to  $\Phi \in \text{Aut}(\mathbf{G})$  then there is some  $u \in \Lambda$  such that*

$$(\text{Inn}_u \circ \Phi)(\mathbf{S}) \subseteq \mathbf{S}.$$

*Proof.* Let  $\mathfrak{u}$  denote the Lie algebra of  $\mathbf{U}$ . As in the proof of Lemma 6.2, the action of  $\Gamma_s$  on  $\mathbf{U}$  induces a linear action of  $\Gamma_s$  on  $\mathfrak{u}$ . Let  $\theta$  be a good unipotent shadow of  $\Gamma_r$ . For each  $\phi \in \text{Aut}(\Gamma)$  and each  $\gamma_s \in \Gamma_s$ , there is some  $\gamma_r \in \text{Fitt}(\Gamma_r)$  so that

$$\phi(0, \gamma_s) = (\gamma_r, \gamma_s).$$

Suppose that  $\Phi \in \text{Aut}(\mathbf{G})$  extends  $\phi$  and  $u \in \mathbf{U}(\mathbb{Q})$  satisfies  $(\text{Inn}_u \circ \Phi)(\mathbf{S}) \subseteq \mathbf{S}$ . Since  $\text{Fitt}(\Gamma_r) \leq \theta$ , it follows that  $u(\gamma_s \cdot u^{-1}) \in \theta$  for all  $\gamma_s \in \Gamma_s$ . The action of  $\Gamma_s$  on  $\mathfrak{u}$  preserves a vector space lattice  $L' \subseteq \mathfrak{u}$  containing  $\log(\theta)$  such that, if  $v = \log(u)$ ,

$$v - \gamma_s \cdot v \in L'$$

for all  $\gamma_s \in \Gamma_s$ . Because  $\mathbf{S}$  is semisimple, the action of  $\Gamma_s$  on  $\mathfrak{u}$  is completely reducible. Applying Lemma 8.8 to each irreducible component of this representation of  $\Gamma_s$ , we may find some vector space lattice  $L \subseteq \mathfrak{u}$  such that  $u \in \mathbf{U}(\mathbb{Q})$  may be chosen such that  $\log(u) \in L$ . Because  $\log(\theta) \subseteq \mathfrak{u}(\mathbb{Q})$ , we may choose  $L \subseteq \mathfrak{u}(\mathbb{Q})$ . Let  $\Lambda \leq \mathbf{U}(\mathbb{Q})$  be any subgroup such that  $\log(\Lambda)$  is a vector space lattice containing  $L$  with finite index. (Such a subgroup exists by the methods of [Seg83, §6B].) The fact that  $\Lambda$  is commensurable with  $\mathbf{U}(\mathbb{Z})$  is immediate from the fact that  $\log(\Lambda) \subseteq \mathfrak{u}(\mathbb{Q})$  is a vector space lattice.  $\square$

Now let

$$A_{\Lambda, \mathbf{H}} = \{ \Phi \in \mathcal{A}_{\mathbf{H}|\mathbf{F}} \mid \Phi(\Lambda) \subseteq \Lambda \}.$$

Then  $A_{\Lambda, \mathbf{H}}$  is commensurable with  $\mathcal{A}_{\mathbf{H}|\mathbf{F}}(\mathbb{Z})$  by [BG06, 8.1], hence is commensurable with  $\text{Aut}(\Gamma_r)$ . Define a  $\mathbb{Q}$ -defined subgroup of  $C(\mathbf{G})$  by

$$C_\Gamma(\mathbf{G}) = \{ (\Phi_R, \Phi_S) \in C(\mathbf{G}) \mid \Phi_R \in \mathcal{A}_{\Gamma_r} \},$$

and

$$A_\Lambda = \{ (\Phi_R, \Phi_S) \in C_\Gamma(\mathbf{G}) \mid \Phi_R \in A_{\Lambda, \mathbf{H}} \text{ and } \Phi_S(\Gamma_s) = \Gamma_s \}.$$

Then  $A_\Lambda$  is commensurable with  $C_\Gamma(\mathbf{G})(\mathbb{Z})$ . Note that the map  $\Theta$  of Lemma 8.6 descends to a map

$$\bar{\Theta} : \mathbf{U} \rtimes C_\Gamma(\mathbf{G}) \rightarrow \text{Aut}(\mathbf{G}),$$

and there is an isomorphism of algebraic groups

$$\mathcal{B} \cong \mathbf{U} \rtimes C_\Gamma(\mathbf{G}) / \ker(\bar{\Theta}).$$

Let

$$\text{Aut}_\Lambda(\Gamma) = \left\{ \phi \in \text{Aut}(\Gamma) \mid \phi|_{\Gamma_r} \in A_{\Lambda, \mathbf{H}} \right\}.$$

Note that  $[\text{Aut}(\Gamma) : \text{Aut}_\Lambda(\Gamma)] < \infty$ . By Lemma 8.9 there is a map

$$\xi : \text{Aut}_\Lambda(\Gamma) \rightarrow \Lambda \rtimes A_\Lambda / \ker(\bar{\theta}).$$

This map is clearly injective, and the preceding discussion shows that its image is of finite index. Therefore the image of  $\text{Aut}(\Gamma)$  in  $\mathcal{B}$  is commensurable with  $\mathcal{B}(\mathbb{Z})$ . This completes the proof.  $\square$

*Remark.* The assumption that our lattice is superrigid in  $S$  cannot be removed. Consider for example  $S = \text{SO}(1, n)$  for  $n \geq 2$  with a lattice  $\Gamma \leq S$  such that  $\Gamma/[\Gamma, \Gamma]$  is infinite. Let  $\tau : \Gamma \rightarrow \mathbb{Z}$  be any nontrivial homomorphism. Then  $\phi_\tau : \mathbb{Z} \times \Gamma \rightarrow \mathbb{Z} \times \Gamma$  defined by

$$\phi_\tau(t, \gamma) = (t + \tau(\gamma), \gamma)$$

is an automorphism of  $\mathbb{Z} \times \Gamma$ , which is a lattice in  $\mathbb{R} \times S$ . However,  $\phi_\tau$  neither is induced by conjugation by an element of  $\mathbb{Q} \subseteq \mathbb{R}$  nor preserves  $S$  in any sense, and  $\phi_\tau$  cannot be extended to an automorphism of  $\mathbb{R} \times S$ .

Automorphisms of the form  $\phi_\tau$  as above are in one-to-one correspondence with elements of  $H^1(\Gamma, \mathbb{Z})$ . If  $\Delta \leq \Gamma$  is a finite index subgroup and  $\sigma \in H^1(\Delta, \mathbb{Z})$ , then  $\phi_\sigma$  defines a partial automorphism of  $\mathbb{Z} \times \Gamma$ . In this way we identify the inverse limit

$$\mathcal{C} = \varprojlim \{ H^1(\Delta, \mathbb{Z}) \mid [\Gamma : \Delta] < \infty \}$$

with a subgroup of  $\text{Comm}(\mathbb{Z} \times \Gamma)$ . Commensurations in  $\mathcal{C}$  do not extend to automorphisms of  $\mathbb{R} \times S$ . For any finite index subgroup  $\Delta \leq \Gamma$ , we may identify  $H^1(\Delta, \mathbb{Q})$  as a subgroup of  $\mathcal{C}$ . In this way, the virtual first rational Betti number of the semisimple quotient of a lattice may be seen as an obstruction to the realization of commensurations as automorphisms of an algebraic group.

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